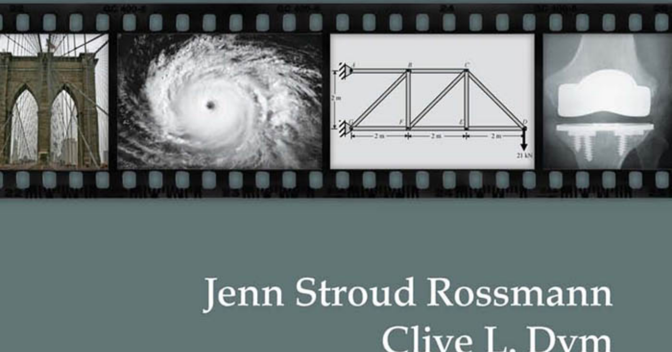
Introduction to

Engineering Mechanics

A Continuum Approach



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Introduction to Engineering Mechanics *A Continuum Approach*

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If science teaches us anything, it's to accept our failures, as well as our successes, with quiet dignity and grace.

Gene Wilder, *Young Frankenstein,* **1974**

Preface

This book is intended to provide a unified introduction to solid and fluid mechanics and to convey the underlying principles of continuum mechanics to undergraduates. We assume that students using this book have taken courses in calculus, physics, and vector analysis. By demonstrating both the connections and the distinctions between solid and fluid mechanics, this book will prepare students for further study in either field or in fields such as bioengineering that blur traditional disciplinary boundaries.

The use of a continuum approach to make connections between solid and fluid mechanics is a perspective typically provided only to advanced undergraduates and graduate students. This book *introduces* the concepts of stress and strain in the continuum context, showing the relationships between solid and fluid behavior and the mathematics that describe them. It is an introductory textbook in strength of materials and in fluid mechanics and also includes the mathematical connective tissue between these fields. We have decided to begin with the *a-ha!* of continuum mechanics rather than requiring students to wait for it.

This approach was first developed at Harvey Mudd College (HMC) for a sophomore-level course called "Continuum Mechanics." The broad, unspecialized engineering program at HMC requires that curriculum planners ask themselves, "What specific knowledge is essential for an engineer who may practice, or continue study, in one of a wide variety of fields?" This course was our answer to the question, what *engineering mechanics* knowledge is essential?

An engineer of any type, we felt, should have an understanding of how materials respond to loading: how solids deform and incur stress; how fluids flow. We conceived of a spectrum of material behavior, with the idealizations of Hookean solids and Newtonian fluids at the extremes. Most modern engineering materials—biological materials, for example—lie between these two extremes, and we believe that students who are aware of the entire spectrum from their first introduction to engineering mechanics will be well prepared to understand this complex middle ground of nonlinearity and viscoelasticity.

Our integrated introduction to the mechanics of solids and fluids has evolved. As initially taught by CLD, the HMC course emphasized the underlying principles from a mathematical, applied mechanics viewpoint. This focus on the structure of elasticity problems made it difficult for students to relate formulation to applications. In subsequent offerings, JSR chose to embed continuum concepts and mathematics into introductory problems, and to build gradually to the strain and stress tensors. We now establish a "continuum checklist"—compatibility [deformation], constitutive law, and equilibrium—that we return to repeatedly. This checklist provides a framework for a wide variety of problems in solid and fluid mechanics.

We make the necessary definitions and present the template for our continuum approach in Chapter 1. In Chapter 2, we introduce strain and stress in one dimension, develop a constitutive law, and apply these concepts to the simple case of an axially loaded bar. In Chapter 3, we extend these concepts to higher dimensions, introducing Poisson's ratio and the strain and stress tensors. In Chapters 4–7 we apply our continuum sense of solid mechanics to problems including torsion, pressure vessels, beams, and columns. In Chapter 8, we make connections between solid and fluid mechanics, introducing properties of fluids and the strain *rate* tensor. Chapter 9 addresses fluid statics. Applications in fluid mechanics are considered in Chapters 10 and 11. We develop the governing equations in both control volume and differential forms. In Chapter 12, we see that the equations for solid *dynamics* strongly resemble those we've used to study fluid dynamics. Throughout, we emphasize real-world design applications. We maintain a continuum "big picture" approach, tempered with worked examples, problems, and a set of case studies.

The six case studies included in this book illustrate important applications of the concepts. In some cases, students' developing understanding of solid and fluid mechanics will help them understand "what went wrong" in famous failures; in others, students will see how the textbook theories can be extended and applied in other fields such as bioengineering. The essence of continuum mechanics, the internal response of materials to external loading, is often obscured by the complex mathematics of its formulation. By building gradually from one-dimensional to two- and three-dimensional formulations and by including these illustrative real-world case studies, we hope to help students develop physical intuition for solid and fluid behavior.

We've written this book for our students, and we hope that reading it is very much like sitting in our classes. We have tried to keep the tone conversational and have included many asides that describe the historical context for the ideas we describe and hints at how some concepts may become even more useful later on.

We are grateful to the students who have helped us refine our approach. We are deeply appreciative of our colleague and friend Lori Bassman (HMC)—of her sense of pure joy in structural mechanics and her ability to communicate that joy. Lori has been a sounding board, contributor of elegant (and fun) homework problems, and defender of the integrity of "second moment of area" despite the authors' stubbornly abiding affection for "moment of inertia." We also thank Joseph A. King (HMC), Harry E. Williams (HMC), Josh Smith (Lafayette), James Ferri (Lafayette), Diane Windham Shaw (Lafayette), Brian Storey (Olin), Borjana Mikic (Smith), and Drew Guswa (Smith). We thank Michael Slaughter and Jonathan Plant, our editors at Taylor & Francis/CRC, and their staff.

We want to convey our warmest gratitude to our families. First are Toby, Leda, and Cleo Rossmann. Thanks especially to Toby, for his direct and indirect support of this project. And then there's Joan Dym, Jordana, and Miriam, and Matt and Ryan and spouses and partners, and a growing number of grandchildren. We are grateful for their support, love, and patience.

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1

Introduction

This textbook, *Introduction to Engineering Mechanics: A Continuum Approach*, is intended to demonstrate the connections between solid and fluid mechanics, and the larger mathematical concepts shared by both fields, while introducing the fundamentals of both solid and fluid engineering mechanics.

Mechanics is the study of the motion or equilibrium of matter and the forces that cause such motion or equilibrium. The reader is likely already familiar with the sort of "billiard ball" mechanics formulated in physics courses—for example, when two such billiard balls collide, applying Newton's second law will help us learn the velocities of both balls after the collision. *Engineering mechanics* mandates that we also consider how the impact will affect the balls: Will they deform or even crack? How many such collisions can they sustain? How does the material chosen for their construction affect both these answers? What design decisions will optimize the strength, cost, or other properties of the balls? Taking a *continuum* approach to engineering mechanics means, essentially, that we will consider what's going on inside the billiard balls and will quantify the *internal response* to external loading.

This book provides an introduction to the mechanics of both solids and fluids and emphasizes both distinctions and connections between these fields. We will see that the material behaviors of ideal solids and fluids are at the far ends of a *spectrum* of material behavior and that many materials of interest to modern engineers—particularly biomaterials—lie between these two extremes, combining elements of both "solid" and "fluid" behavior.

Our objectives are to learn how to formulate problems in mechanics and how to reduce vague questions and ideas into precise mathematical statements. The floor of a building may be strong enough to support us, our furniture, and even the occasional fatiguing dance party, but if not designed carefully, the floor may deflect considerably and sag. By learning how to predict the effects of forces, stresses, and strains, we will become better designers and better engineers.

1.1 A Motivating Example: Remodeling an Underwater Structure

Underwater rigs like that shown in Figure 1.1 are commonly used by the petroleum industry to harvest offshore oil. Over the life of a structure, many sea creatures and plants attach themselves to the supports. When wells have dried up, the underwater structures can be removed in manageable segments and towed to shore. However, this process results in the loss of both the reef dwellers attached to the platform's trusses and the larger fish who feed there. Corporations often abandon their rigs rather than incurring the financial and environmental expense of removal. An engineering firm would like to make use of a decommissioned rig by remodeling it as an artificial reef, providing a hospitable sea habitat. This firm must find ways to strengthen the supports and to affix the reef components to sustain sea life.

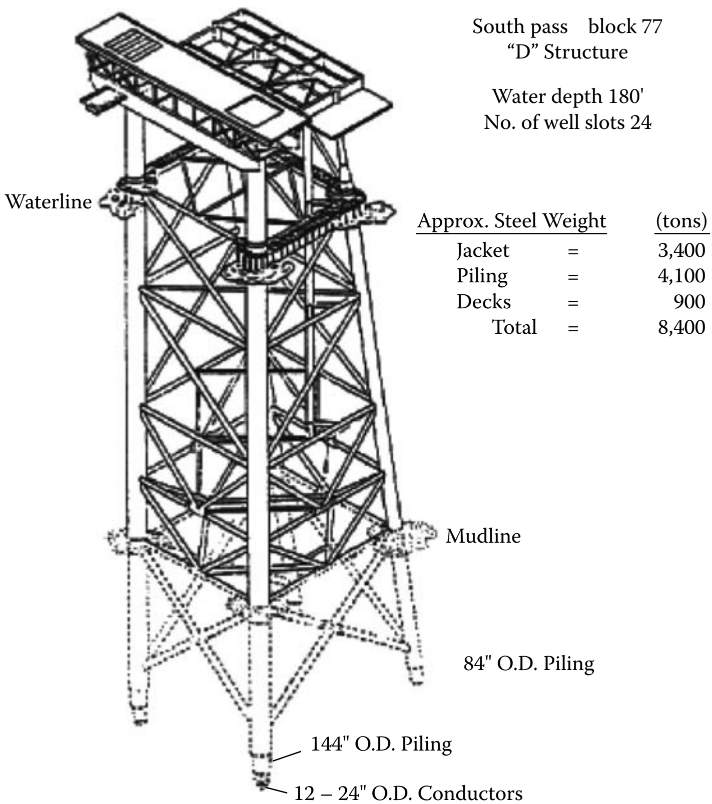


FIGURE 1.1
Mud-slide-type platform. (From the Committee on Techniques for Removing Fixed Offshore Structures and the Marine Board Commission on Engineering and Technical Systems, National Research Council, An Assessment of Techniques for Removing Offshore Structures, Washington, DC: National Academy Press, 1996. With permission.)

The rig support structure was initially designed to support the drilling platform above the water level. As the oil drill itself was mobile, the structure was built so that it could remain balanced, without listing, under this dynamic loading. In its new life as the support for an artificial reef, this structure must continue to withstand the weight of the platform and the changing loads of wind and sea currents, and it must also support the additional loading of concrete "reef balls" and other reef-mimicking assemblies (Figure 1.2), as well as the weight of the reef dwellers.

To remodel the underwater rig, a team of engineers must dive below the water surface to attach the necessary reef balls and other attachments. The reef balls themselves may be lowered using a crane. A conceptualization of this is shown in Figure 1.3.

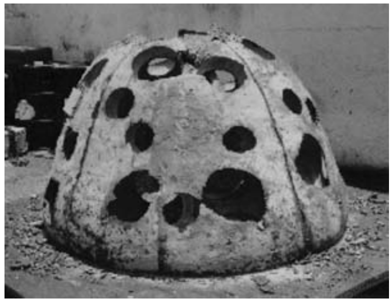


FIGURE 1.2
Concrete reef ball. (Courtesy of the Reef Ball Foundation, Athens, GA.)

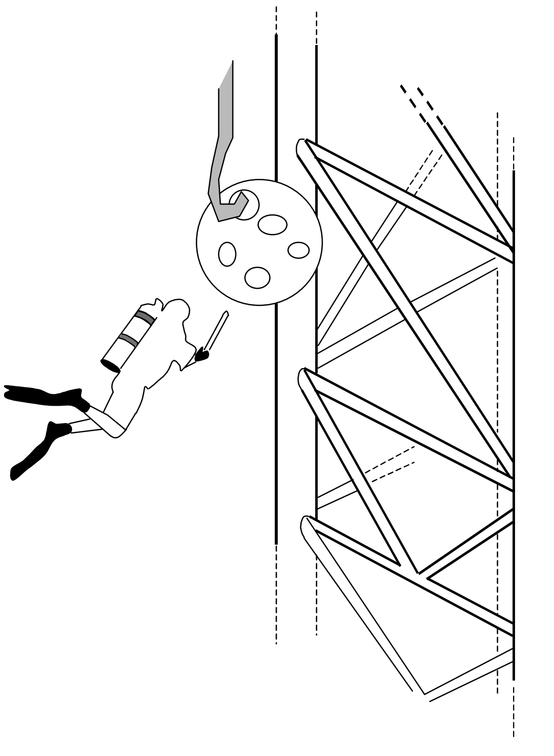


FIGURE 1.3
Rendering of scuba diver at work remodeling underwater rig structure.

Among the factors that must be considered in the redesign process is the structural performance of the modified structure, its ability to withstand the required loading. An additional challenge to the engineering firm is the undersea location of the structure. What materials should be chosen so that the structure remains sound? How should the additional supports and reef assemblies be added? What precautions must engineers and fabricators take when they work underwater? What effects will the exposure to the ocean environment have on their structure, equipment, and bodies? We address many of these issues in this book. Throughout, we return to this problem to demonstrate the utility of various theoretical results, and we rely on first principles that look familiar.

1.2 Newton's Laws: The First Principles of Mechanics

Newton's laws provide us with the *first principles* that, along with conservation equations, guide the work we do in continuum mechanics. Many of the equations we use in problem solving are directly descended from these elegant statements. These laws were formulated by Sir Isaac Newton (1642–1727), based on his own experimental work and on the observations of others, including Galileo Galilei (1564–1642). Newton's laws are expressed as follows:

> Newton's first law: A body remains at rest or moves in a straight line with constant velocity if there is no unbalanced force acting on it.
>
> Newton's second law: The time rate of change of momentum of a body is equal to (and in the same direction as) the resultant of the forces acting on it:
>
> $$\sum \underline{F} = \frac{d}{dt}(m\ \underline{V}). \tag{1.1}$$
>
> When the mass of the body of interest is constant, this has the form
>
> $$\Sigma\ \underline{F} = m\ \underline{a}, \tag{1.2}$$
>
> and when $\underline{a} = \underline{0}$, this means that we have
>
> $$\Sigma\ F = 0. \tag{1.3}$$
>
> (This last class of problems is often called "statics.")
>
> Newton's third law: To every action there is an equal and opposite reaction. That is, the forces of action and reaction between interacting bodies are equal in magnitude and exactly opposite in direction.

Forces always occur, according to Newton's third law, in pairs of equal and opposite forces. The downward force exerted on the desk by your pencil is accompanied by an upward force of equal magnitude exerted on your pencil by the desk.

1.3 Equilibrium

We have alluded to the concept of equilibrium (also known as *static* equilibrium) in our discussion of Newton's second law. To be in equilibrium, a three-dimensional object must satisfy six equations. In Cartesian coordinates, these are as follows:

$$\sum F_x = 0$$
$$\sum F_y = 0$$
$$\sum F_z = 0 \tag{1.4a}$$

$$\sum M_x = 0$$
$$\sum M_y = 0$$
$$\sum M_z = 0 \tag{1.4b}$$

These equations can be written more concisely in vector form as

$$\Sigma \underline{F} = \underline{0} \tag{1.5}$$

$$\Sigma \underline{M} = \underline{0}, \tag{1.6}$$

and represent the statements "the sum of forces equals zero" and "the sum of moments (about some reference axis) equals zero." One advantage of writing these equations in vector form is that we don't have to specify a coordinate system!

For planar (two-dimensional) situations or models, equilibrium requires the satisfaction of only three equations, usually

$$\sum F_x = 0 \tag{1.7a}$$

$$\sum F_y = 0 \tag{1.7b}$$

$$\sum F_z = 0 \tag{1.7c}$$

These equations essentially state that the object is neither translating (in the x or y directions) nor rotating (about the z axis) in the xy plane as a result of applied forces.

It is useful to distinguish between forces that act externally and those that act internally. *External* loads are applied to a structure by, for example, gravity or wind. Reaction forces are also external: They occur at supports and at points where the structure is prevented from moving in response to the external loads. These supports may be surfaces, rollers, hinges; fixed or free. *Internal* forces, on the other hand, *result* from the applied external loads and are what we are concerned with when we study continuum mechanics. These are forces that act within a body as a result of all external forces. Chapter 2 shows how the principle of equilibrium helps us calculate these internal forces.

1.4 Definition of a Continuum

In elementary physics, we concerned ourselves with particles and bodies that behaved like inert billiard balls, bouncing off each other and interacting without deformation or other changes. In continuum mechanics, we consider the effects of deformation, of internal forces within bodies, to get a fuller sense of how bodies react to external forces.

We would like to be able to consider these bodies as whole entities and not have to account for each individual particle composing each body. It would be much more convenient for us to treat the properties (e.g., density, momentum, forces) of such bodies as continuous functions. We may do this if the body in question is a *continuum*.

We may treat a body as a continuum if the ensemble of particles making up the body acts like a continuum. We can then consider the average or "bulk" properties of the body and can neglect the details of the individual particle dynamics. Acting like a continuum means that no matter how small a chunk of the body we consider, the chunk will have the same properties (e.g., density) as the bulk material.

Mathematically, we define a continuum as a continuous distribution of matter in space and time. For a mass m_n contained in a small volume of space, V_n, surrounding a point *P*, as in Figure 1.4, we can define a mass density ρ:

$$\rho(P)=\lim_{\substack{n\to\infty \\ V_n\to 0}}\frac{m_n}{V_n}. \tag{1.8}$$

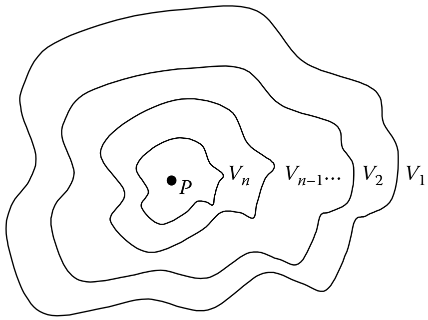


FIGURE 1.4
Volumes V_i surrounding point *P*.

So, a material continuum is a material for which density (of mass, momentum, or energy) exists in a mathematical sense. We are able to define its properties as continuous functions and to neglect what's happening on the microscopic, molecular level in favor of the macroscopic, bulk behaviors.

Note that if V_n truly goes to zero, gases and liquids will not satisfy this equation: Density will be undefined. (If the volume goes to zero, it will not have a chance to enclose any atoms—so naturally, the density will be undefined!) Yet we still think of these materials as continua. So physically, our definition of a continuum is a material for which

$$\left|\rho-\frac{m_n}{V_n}\right|<\varepsilon \text{ as } n\to\infty. \tag{1.9}$$

Here, ϵ represents a *very* small number approaching zero, indicating that the mathematical definition of density approaches a usable value, ρ.

Sometimes it is easier to get a grasp on what is not a continuum than on what is. Almost all solids satisfy the definition handily. Solids are generally much denser than fluids. For fluids, it can be harder to pin down a "density" once gas molecules get sparse. Interstellar space, for example, where the objects of interest (e.g., planets, asteroids) are not much farther apart than the molecules of the interstellar medium, is surely stretching the limits of the definition of a continuum. Fortunately, another test for continuity is available. It's especially applicable to fluids.

A given material may be called a *continuum* if the Knudsen[1] number, *Kn,* is less than about 0.1. The Knudsen number is defined as

$$Kn=\frac{\lambda}{L}, \tag{1.10}$$

where L is a problem-specific characteristic length, such as a diameter or width, and λ is the material's "mean free path," or average distance between particle collisions, obtainable from

$$\lambda = 0.225 \frac{m}{\rho d^2}, \tag{1.11}$$

where m is the mass of a molecule, ρ is its density, and d is the diameter of a molecule. For example, for air $m = 4.8 \times 10^{-26}$ kg, $d = 3.7 \times 10^{-10}$ m, and at atmospheric conditions λ is approximately 6×10^{-6} cm; at an altitude of 100 km it is 10 cm, and at 160 km it's 5000 cm. So at higher altitudes, the continuum assumption is unacceptable and the molecular dynamics must be considered in the governing equations.

The ease with which we can define density, and continuity, is not the only difference between solids and fluids:

> A *solid* is a three-dimensional continuum that supports both tensile and shear forces and stresses. The atoms making up a solid have a fixed spatial arrangement—often a crystal lattice structure—in which atoms are able to vibrate and spin and their electrons can fly and dance around but the microstructure is fixed. Because of this, although it's possible to distort or destroy the shape taken by a solid, it is generally said that a solid object retains its own shape. For solids, we will be able to relate *stresses* and *strains* by a *constitutive law.*
>
> A *fluid* may be a liquid or a gas. A fluid, it's been said, is something that flows: Liquids assume the shape of their containers, and gases expand to fill any container. This is because the atoms comprising a fluid are not spatially constrained like those of a solid. More formally, a fluid is a three-dimensional continuum that (a) cannot support tensile forces or stresses, and (b) deforms continuously under the smallest shearing forces or stresses. For fluids, we will be able to relate stresses and strain rates by a constitutive law.

We note that the distinction between solid and fluid behavior is not always clear-cut; there are classes of materials whose behavior situates them in a sort of middle ground. We explore this middle ground further in Case Study 5. The existence of this middle ground provides us with more motivation to understand the broad field of continuum mechanics and the connections between solid and fluid behavior.

In this text we are interested in how Newton's laws apply to continua. Some of the relevant consequences of Newton's laws, which we discuss in more detail later, are as follows:

- Momentum is always conserved, in both solids and fluids. *Equilibrium* equations (see Section 1.3) are the mathematical expressions of the conservation of momentum.
- Equilibrium must apply both to entire bodies and to sections of, or particles within, those bodies. This is one of the reasons why free-body diagrams (FBDs) are so valuable: They illustrate the equilibrium of a section of a larger body or system. This is also why we use *control volumes* to analyze fluid flows.
- Mass is conserved.
- Area is a vector, having both magnitude (size) and direction, which is defined by a unit vector normal to the area and directed outward from the free body or volume of interest.
- Forces produce changes in shape and geometry, which are characterized in terms of *strains* for solids and *strain rates* for fluids.

In the real world, material objects are subjected to *body* forces (e.g., gravitational and electromagnetic forces), which do not require direct contact, and *surface* forces (e.g., atmospheric pressure, wind and rain, burdens to be carried), which do. We want to know how the material in the body reacts to external forces. To do this, we will need to (1) characterize the deformation of a continuous material, (2) define the *internal* loading, (3) relate this to the body's deformation, and (4) make sure that the body is in equilibrium. This is what continuum mechanics is all about.

1.5 Mathematical Basics: Scalars and Vectors

The familiar distinction between scalars and vectors is that a vector, unlike a scalar, has direction as well as magnitude. Examples of scalar quantities are time, volume, density, speed, energy, and mass. Velocity, acceleration, force, and momentum are vectors and contain the extra directional information. We typically denote vectors with a bold font or an underline. This book underlines all vectors.

A vector $\underline{V}$ may be expressed mathematically by multiplying its magnitude, $\underline{V}$, by a unit vector $\underline{n}$ (note: $|\ \underline{n}\ | = 1$, and $\underline{n}$'s direction coincides with $\underline{V}$):

$$\underline{V} = V\underline{n}. \tag{1.12}$$

We may also write a vector $\underline{V}$ in terms of its *components* along the primary directions, whether these are the Cartesian (x, y, z) directions or cylindrical (r, θ, z) or another set. In Cartesian coordinates this is simply written as

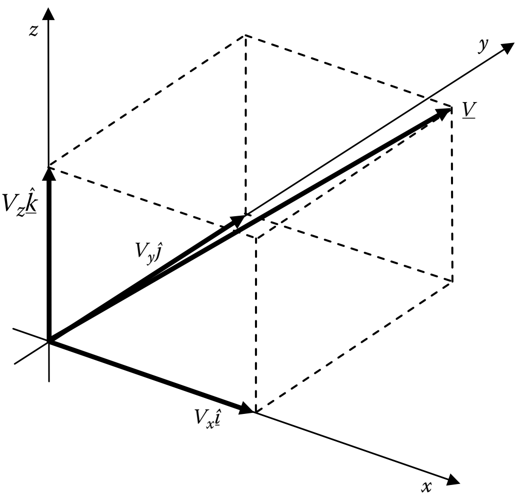


FIGURE 1.5
Decomposition of vector $\underline{V}$ in x, y, z coordinates.

$$\underline{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k} \tag{1.13}$$

based on a situation like that shown in Figure 1.5. In general, in coordinates (x_1, x_2, x_3) with unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$, we will be able to write any vector $\underline{V}$ as

$$\underline{V} = V_1\hat{e}_1 + V_2\hat{e}_2 + V_3\hat{e}_3 \tag{1.14}$$

or as (V_1, V_2, V_3)—what we called a column vector in linear algebra.[2] We remember that the magnitude of $\underline{V}$ can be obtained:

$$V = |\,\underline{V}\,| = \sqrt{(V_1^{\,2} + V_2^{\,2} + V_3^{\,2})}, \tag{1.15}$$

so $\underline{V} = \underline{0}$ if, and only if, $V_1 = V_2 = V_3 = 0$.

The calculated dot and cross products are also of interest. Remember that the result of taking a dot product is a *scalar* and that the result of a cross product is a *vector*. Briefly,

$$\underline{u} \bullet \underline{v} = |\,\underline{u}\,|\;|\,\underline{v}\,| \cos\theta, \tag{1.16}$$

where θ is the angle between vectors $\underline{u}$ and $\underline{v}$, and $0 \le \theta \le \pi$. Physically, the scalar or *dot product* can be thought of as the magnitude of $\underline{u}$ times the component of $\underline{v}$ along $\underline{u}$. In terms of components,

$$\underline{u} = u_1 \hat{e}_1 + u_2 \hat{e}_2 + u_3 \hat{e}_3 \,, \tag{1.17}$$

$$\underline{v} = v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3 \,, \tag{1.18}$$

$$\underline{u} \bullet \underline{v} = u_1 v_1 + u_2 v_2 + u_3 v_3. \tag{1.19}$$

Also, the cross product results in a vector that is perpendicular to both $\underline{u}$ and $\underline{v}$:

$$\underline{u} \times \underline{v} = \underline{w}, \tag{1.20}$$

where

$$| \; \underline{w} \; | = | \; \underline{u} \; | \; | \; \underline{v} \; | \sin\theta, \tag{1.21}$$

and

$$\underline{u} \times \underline{v} = (u_2 v_3 - u_3 v_2) \; \hat{e}_1 + (u_3 v_1 - u_1 v_3) \; \hat{e}_2 + (u_1 v_2 - u_2 v_1) \; \hat{e}_3 \,. \tag{1.22}$$

We notice that this has the form of a determinant:

$$\underline{u} \times \underline{v} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}. \tag{1.23}$$

When we work with vectors, we may find ourselves getting stuck carrying around a set of variables, $x_1, x_2, \ldots x_n$. This can become unwieldy, and so we may use a shortcut known as *index notation*. Using this shortcut, we write x_i, $i = 1, 2, \ldots n$, and call i the *index*. If, for example, we are working with the equation

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = p, \tag{1.24}$$

we may write this as

$$\sum_{i=1}^{3} a_i x_i = p \tag{1.25}$$

and may further simplify life by writing

$$a_i x_i = p. \tag{1.26}$$

This substantially more efficient shortcut is known as the *summation convention*: The repetition of the index represents summation with respect to that index over its range. Using index notation and the summation convention, we could rewrite the definition of dot product (1.19) as

$$\underline{u} \cdot \underline{v} = u_i v_i \ . \tag{1.27}$$

We understand scalars to contain the least possible amount of information—only a magnitude—while a vector contains more information and can be manipulated in more ways. The curious student may be wondering whether there is any type of variable that can contain more information than a vector. That provocative question is answered in Chapter 3.

1.6 Problem Solving

Any reader of your solution to a given problem should be able to follow the reasoning behind it. To test yourself you may find a stranger on the street and ask whether your logic is clear, or you may simply make sure that you have included each of the following steps:

1. State what is given: The speed of major league fastball and distance from pitcher's mound to home plate, 60 feet 6 inches are given.
2. State what is sought: Find the time a batter has to react to an incoming pitch.
3. Draw relevant sketches or pictures: In particular, isolate the body (or relevant control volume) to see the forces involved, by means of a free-body diagram.
4. Identify the governing principles (e.g., Newton's second law).
5. Calculations: Keep in symbolic form (e.g., $v = d/t$).
6. Check the physical dimensions of your answer: Will answer have dimensions of time? If it looks like it will be a length, go back.
7. Complete calculations: Substitute in numbers; wait as long as possible before plugging in numbers. This gives you time to do a dimensional check and to think about whether the dependencies you've found make sense (should the answer depend on the pitcher's wingspan?) and allows you to reuse the model for similar problems that may arise.
8. State answers and conclusions.

In the worked example problems that follow each chapter in this textbook, these steps are followed.

1.7 Examples

Example 1.1

A force $\underline{F}$ with magnitude 100 N passes through the points (1, 2, 1) and (3, –2, 2) (pointing toward (3, –2, 2)) where coordinates are in meters. Determine the following:

(a) The magnitudes of the *x*, *y*, and *z* scalar components of $\underline{F}$
(b) The moment of $\underline{F}$ about the origin
(c) The moment of $\underline{F}$ about the point (2, 0.3, 1)

Given: Force vector.

Find: Components of vector and moment of vector about two points.

Assume: No assumptions are necessary.

Solution

We can obtain a solution using either a holistic "vector approach" or a piece-by-piece "component approach." We will demonstrate both approaches.

Vector Approach

(a) The force can be written as $\underline{F} = F\,\underline{n}$ where $\underline{n}$ is the unit vector in the direction of the force:

$$\underline{n} = \frac{2\underline{\hat{i}} - 4\underline{\hat{j}} + 1\underline{\hat{k}}}{\sqrt{2^2 + (-4)^2 + 1^2}} = 0.436\underline{\hat{i}} - 0.873\underline{\hat{j}} + 0.218\underline{\hat{k}}$$

$$\underline{F} = 100\,\underline{n} = 43.6\,\underline{\hat{i}} - 87.3\,\underline{\hat{j}} + 21.8\,\underline{\hat{k}}\ \text{N}$$

so the scalar components of $\underline{F}$ are $Fx = 43.6$ N, $Fy = -87.3$ N, and $Fz = 21.8$ N.

(b) The moment of F about the origin is found using $\underline{Mo} = \underline{r} \times \underline{F}$, where $\underline{r}$ is a vector from the origin to any point on the line of action of $\underline{F}$.

Using $\underline{r} = 1\,\underline{\hat{i}} + 2\,\underline{\hat{j}} + 1\,\underline{\hat{k}}$, $\underline{r} \times \underline{F}$ may be written as a determinant:

$$\underline{M}_o = \begin{vmatrix} \underline{\hat{i}} & \underline{\hat{j}} & \underline{\hat{k}} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \underline{\hat{i}} & \underline{\hat{j}} & \underline{\hat{k}} \\ 1 & 2 & 1 \\ 43.6 & -87.3 & 21.8 \end{vmatrix}$$

$$= [2\,(21.8) - 1\,(-87.3)]\,\underline{\hat{i}} - [1\,(21.8) - 1\,(43.6)]\,\underline{\hat{j}} + [1\,(-87.3) - 2\,(43.6)]\,\underline{\hat{k}}$$

$$\underline{M}_o = 130.9\,\underline{\hat{i}} + 21.8\,\underline{\hat{j}} - 174.5\,\underline{\hat{k}}\ \text{N·m.}$$

(c) A vector $\underline{r}$ is needed from the point P (2, 0.3, 1) to any point on the line of action of $\underline{F}$. We see that $\underline{r} = -1\,\underline{\hat{i}} + 1.7\,\underline{\hat{j}} + 0\,\underline{\hat{k}}$ is such a vector (goes to the point (1, 2, 1)). Then $\underline{M}_p = \underline{r} \times \underline{F}$:

$$\underline{M}_p = \begin{vmatrix} \underline{\hat{i}} & \underline{\hat{j}} & \underline{\hat{k}} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \underline{\hat{i}} & \underline{\hat{j}} & \underline{\hat{k}} \\ -1 & 1.7 & 0 \\ 43.6 & -87.3 & 21.8 \end{vmatrix}$$

$$= [1.7\,(21.8) - 0]\,\underline{\hat{i}} - [-1\,(21.8) - 0]\,\underline{\hat{j}} + [-1\,(-87.3) - 1.7\,(43.6)]\,\underline{\hat{k}}$$

$$\underline{M}_p = 37.1\,\underline{\hat{i}} + 21.8\,\underline{\hat{j}} + 13.2\,\underline{\hat{k}}\ \text{N·m.}$$

Scalar (Components) Approach

(a) The length of the segment from (1, 2, 1) to (3, –2, 2) is

$$\sqrt{(3-1)^2 + (-2-2)^2 + (2-1)^2} = \sqrt{22+(-4)2+12} = \sqrt{21}$$

v

Direction Cosines	**Then**
$l = 2/\sqrt{21} = 0.436$	$F_x = 100\ (0.436) = 43.6$ N
$m = -4/\sqrt{21} = -0.873$	$F_y = 100\ (-0.873) = -87.3$ N
$n = 1/\sqrt{21} = 0.218$	$F_z = 100\ (0.218) = 21.8$ N

(b) Remember that we can consider the force $\underline{F}$ to be acting at any point along its line of action. Choosing (1, 2, 1), the moments about the x, y, and z axes through the origin are

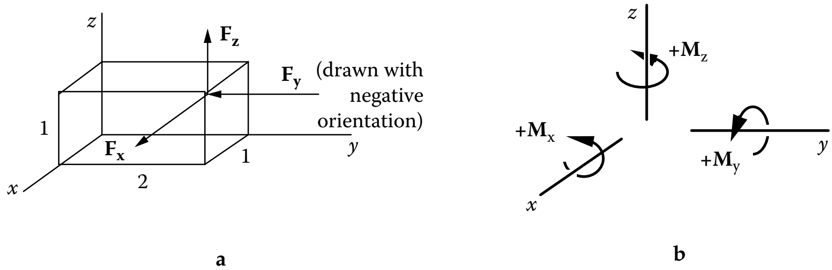


FIGURE 1.6

$$M_{ox} = 1\ (87.3) + 2\ (21.8) = 130.9\ \text{N·m}.$$

(F_x is parallel to the x axis and thus does not have a moment about the x axis.)

$$M_{oy} = 1\ (43.6) - 1\ (21.8) = 21.8\ \text{N·m}$$

$$M_{oz} = -2\ (43.6) - 1\ (87.3) = -174.5\ \text{N·m}.$$

(c) Use the same procedure as part (b). In this case, the distances required are from the point of action of the force (choose (1, 2, 1) as previously) to the point P (2, 0.3, 1):

$$M_{px} = (1 - 1)\ (87.3) + (2 - 0.3)\ (21.8) = 37.1\ \text{N·m},$$

$$M_{py} = (1 - 1)\ (43.6) + (2 - 1)\ (21.8) = 21.8\ \text{N·m},$$

$$M_{pz} = -\ (2 - 0.3)\ (43.6) + (2 - 1)\ (87.3) = 13.2\ \text{N·m}.$$

Example 1.2

A clever sophomore wants to weigh himself but has access only to a scale (A) with capacity limited to 500 N and a small 80 N spring dynamometer

(B). With the rig shown he discovers that when he exerts a pull on the rope so that B registers 76 N, the scale reads 454 N. What are his correct weight and mass?

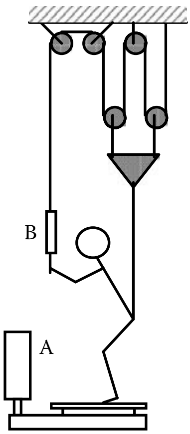


FIGURE 1.7

Given: Geometry of problem, weight indicated on scale A.
Find: True weight and mass of student.
Assume: No assumptions are necessary.

Solution

We assume the tension in the continuous top rope is constant, and we'll neglect the mass of the pulleys. The relevant free-body diagrams are (the circles are the lower pulleys):

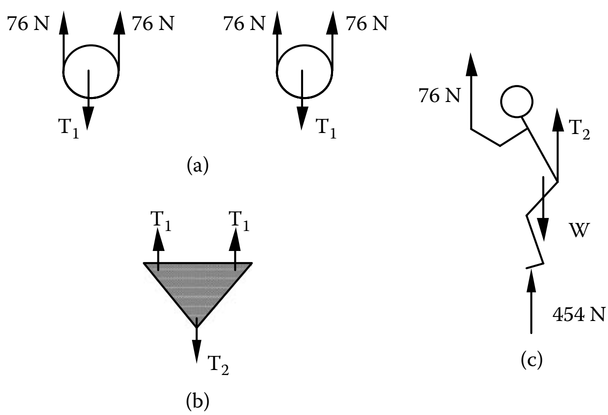


FIGURE 1.8

Next, we ensure that $\Sigma Fy = 0$ holds for each FBD—that is, that each part is in equilibrium.

From diagram (a),

$$T_1 = 76\text{ N} + 76\text{ N} = 152\text{ N}.$$

From diagram (b),

$$T_2 = T_1 + T_1 = 304\text{ N}.$$

From diagram (c),

$$W = 454\text{ N} + 76\text{ N} + T_2 = 834\text{ N}.$$

So, his mass is

$$\frac{834\text{ N}}{9.81\text{ m/s}^2} = 85.0\text{ kg}.$$

1.8 Problems

1.1 The premixed concrete in a cement truck can be treated as a fluid continuum when it is poured into a mold. Sand flowing from a large bucket can also be considered a fluid. Describe three other examples in which an aggregate of solid objects flows likes a fluid continuum.

1.2 Investigate the reef balls used in creating artificial reef environments. What parameters are most important to the successful maintenance of a stable marine environment?

1.3 Find the angle θ between the two vectors $\underline{F}_1 = 4\hat{i} + 3\hat{k}$ and $\underline{F}_2 = \hat{i} + 7\hat{k}$ using their dot product.

1.4 Find and sketch the cross product $\underline{F}_1 \times \underline{F}_2$, given $\underline{F}_1 = -5\hat{i} + 3\hat{k}$ and $\underline{F}_2 = \hat{i} - 4\hat{k}$.

1.5 Determine the force F and the angle θ required to keep the pulley system shown in static equilibrium.

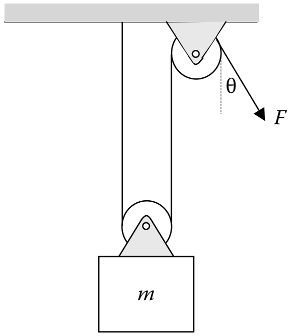


FIGURE 1.9

1.6 A force F acts on a uniform pendulum as shown. Find the reaction forces at the pin connection and the angle θ, letting $F = 100$ N, $d = 1.6$ m, and $W = 300$ N.

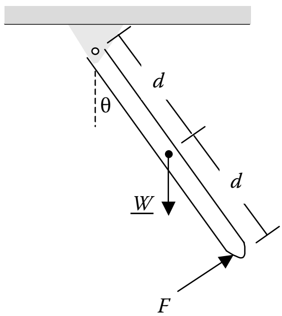


FIGURE 1.10

Notes

1. The Knudsen number is named for Martin Hans Christian Knudsen (1871–1949), professor at the University of Copenhagen and author of *The Kinetic Theory of Gases* (London, 1934). In physical gas dynamics, the Knudsen number defines the extent to which a gas behaves like a collection of independent particles (Kn >>1) or like a viscous fluid (Kn <<1).
2. We have written the column vector of V's components as a row vector to save space.

2

Strain and Stress in One Dimension

In the previous chapter, we stated that in order to study continuum mechanics—that is, to characterize the response of a continuous material to external loading—we must (1) characterize the material's deformation, (2) define its internal loading and (3) relate this to its deformation, and (4) ensure that the body is in equilibrium.[1] We begin this chapter by considering the deformation of a material under loading.

Returning to our example of the remodeling of an underwater oil rig as an artificial reef, we want to examine the trusses of the existing rig. As we have seen (Figure 1.1 and Figure 1.3), the rig is composed of many slender steel members that must withstand the cyclic loading of ocean currents as well as other loads. Each member may be pulled or pushed along its axis, as in Figure 2.1, and by isolating each member we can begin to determine whether the members can withstand this loading.

This raises the question of what it means to "withstand" a load. Is it sufficient for the member to sustain the load without incurring damage or breaking, or is it necessary for it to sustain the load without deforming or bending?

You may have noticed that a standard office table or desk can support far more weight or force than it does when serving as a writing table or computer desk and that some chairs can support the weight of several people without breaking. These are not examples of wasteful or inefficient designs. In fact, these products have been designed for *stiffness* rather than for *strength*. Instead of merely building a chair strong enough to hold the average person, designers have chosen to make the chair stiff enough that its deflections can be limited to some small amount, under a load much larger than it is expected to typically carry. Under normal use, therefore, the chair should not deflect perceptibly. Designing for stiffness means minimizing or limiting deflections and is generally a much more restrictive proposition than designing purely for strength. In this chapter, we discover ways to characterize the stiffness and strength of materials and structures.

To begin to design for stiffness by minimizing deflection, we must understand how to characterize the deformation a loaded body will undergo.

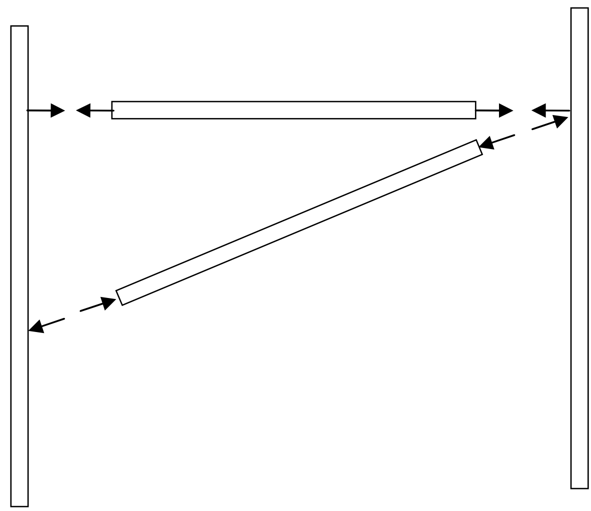


FIGURE 2.1
Isolated members of underwater structure.

2.1 Kinematics: Strain

In continuum mechanics, we want to characterize how bodies respond to the effects of external loading and how these responses are distributed through the bodies. One way a body responds to external loads is by deforming. We develop a way of quantifying its deformation relative to its initial size and shape, and we call this relative deformation *strain*.

2.1.1 Normal Strain

When an axial force is applied to a body, the distance between any two points A and B along the body changes. We call the initial, undeformed length between two points A and B the *gage length* (or *gauge length)*. During a tensile experiment such as the one sketched in Figure 2.2, we may measure the change in gage length as a function of applied force. What interests us is how much this gage length changes, relative to its initial value—in other words, the *intensity of deformation*.

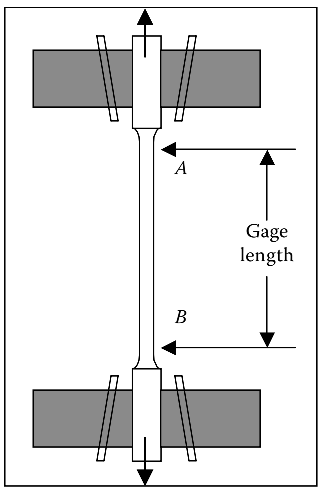


FIGURE 2.2
Tension specimen.

In Figure 2.2, the bar is acted on, or loaded, at its ends by two equal and opposite axial forces. (An axial force is one that coincides with the longitudinal axis of the bar and acts through the centroid of the cross section.) These forces, called *tensile* forces, tend to stretch or elongate the bar. We say that such a bar is in tension.

If L_o is the initial gage length and L is the observed length of the same segment under an applied load, the gage elongation is $\Delta L = L - L_o$. The elongation ε per unit of initial gage length, or "deformation intensity," is then

$$\varepsilon = \frac{L - L_o}{L_o} = \frac{\Delta L}{L_o}. \tag{2.1}$$

This expression for epsilon defines the macroscopic extensional strain.

It is also possible for this apparatus to load a bar with two equal and opposite forces directed toward each other, as in the sketch in Figure 2.3. These forces, called *compressive* forces, tend to shorten or compress the bar. We say that such a bar is in compression. Note that for compressive loading, $\Delta L < 0$, and the normal strain is negative.

Both tensile (tending to elongate) and compressive (tending to shorten) deformations result in *normal* strain, defined as the change in length of our material relative to its initial undeformed length. Normal strain is a dimensionless quantity but is often represented as having dimensions of length/length, in./in., m/m, or mm/mm. Sometimes it is given as a percentage.

In some applications, we use a slightly more careful definition of strain. This is sometimes called the *natural* or *true* strain as distinct from the *engineering* strain defined by equation (2.1). In this true strain definition, a strain increment dε is integrated over the bar:

$$\bar{\varepsilon} = \int_{L_o}^{L} d\varepsilon = \int_{L_o}^{L} \frac{dL}{L} = \ln\left(\frac{L}{L_o}\right) = \ln(1+\varepsilon). \tag{2.2}$$

For very small strains, this natural strain is coincident with the engineering normal strain ε.

In a third definition of strain, we consider that each and every planar section normal to the axis moves a uniform (over the plane) distance along the axis, $u(x)$. An element of the axis that was originally of length dx is thus

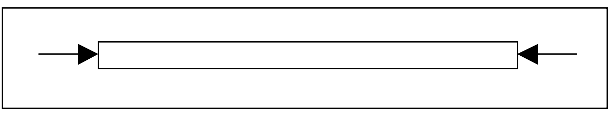


FIGURE 2.3
Bar in compression.

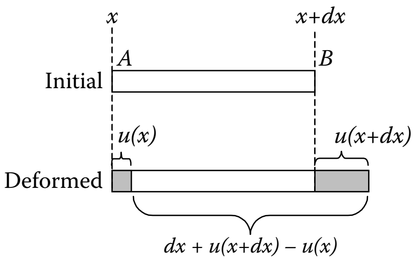


FIGURE 2.4
One-dimensional stretching of a bar.

stretched to a new length, $dx + u(x + dx) - u(x)$. This is illustrated in Figure 2.4.

For this deformation we define strain—in the same spirit as the first definition—as in equation (2.3).

$$\varepsilon = \frac{\text{change in length}}{\text{original length}} = \frac{\left[dx + u(x+dx) - u(x)\right] - dx}{dx}, \tag{2.3}$$

or, retaining only the first-order term in a Taylor series expansion of $u(x + dx)$, we find

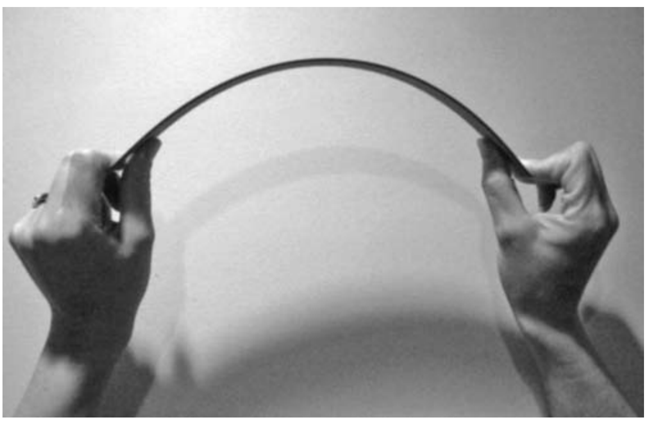
$$\varepsilon \cong \frac{[u(x) + u'(x)dx - u(x)]}{dx} = u'(x) = \frac{du}{dx}. \tag{2.4}$$

In Section 2.7, we use equation (2.4) to express equilibrium in terms of the displacement $u(x)$, to illustrate where compatibility is applied, and to obtain a classic result for the extension of an axially loaded bar.

We also note that the quantity $[u(x + dx) - u(x)]$ represents the *relative displacement* of point B (at $x + dx$) with respect to point A (at x). This will provide a useful context for a more general definition of strain that we develop in Section 3.3.

Example

By bending a thin ruler, you are able to deform it into a circular arc. This arc, with a radius of 30 in., encloses an angle of 23° at center, as shown. Find the average normal strain developed in the ruler.



Given the initial length of the ruler, L_o, which we assume to be exactly 12 in., and the characteristics of a circular arc formed when it is deformed under bending, we must find the intensity of deformation, or induced strain. Since we know that strain is a measure of the change in a body's length relative to the original length, we must determine how much the ruler's length of 12 in. changes under this deformation.

Recalling that the arc length of a circular arc is given by the equation

$$\textit{arc length} = r\theta$$

and that in this case, the arc length is the deformed length of the ruler, L, we have

$$L = r\theta = (30 \text{ in.}) \cdot 23° \cdot \frac{2\pi \text{ rad}}{260°}$$

$$= (30 \text{ in.}) \cdot (0.4014 \text{ rad}) = 12.04277 \text{ in.}$$

Normal strain is then calculated

$$\varepsilon = \frac{\text{change in length}}{\text{original length}} = \frac{L - L_o}{L_o} = \frac{0.04277 \text{ in.}}{12 \text{ in.}} = 0.003564 \frac{\text{in.}}{\text{in.}}$$

For convenience, such a small strain might be reported as 3564 micro-inches per inch (μin./in.), or 3564 *microstrain*, or alternatively as a 0.36% strain.

2.1.2 Shear Strain

Bodies may experience both normal and *shear* deformations and, hence, normal and shear strains. When an axial tensile load is applied to a body, it causes a longitudinal tensile deformation: an elongation. Similarly, an axial compressive load will cause a longitudinal compressive deformation: a shortening. When a shear force is applied to a body, it will cause an angular deformation.

To visualize the effect of shear strain, consider a motor mount as shown in Figure 2.5a. The motor mount is composed of a block of elastic material (our "body") with attachments to allow for connection to the base of the motor and the support structure. A force P is applied at the top of the block. This subjects the block/body (of initial height L) to a pair of shear forces, as shown in Figure 2.5b. If we imagine that the block is composed of many thin layers and that each layer will slide slightly with respect to its neighbor, we may visualize how the angular distortion of the block will develop.

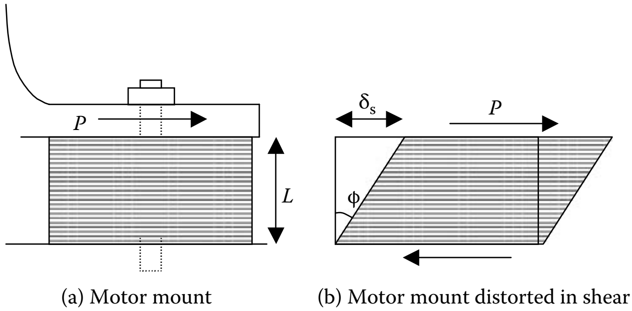


FIGURE 2.5
Shear strain.

As for normal strain, several definitions of shear strain exist. The *engineering* shear deformation incurred is ϕ, the *change* in an initially right angle. This is the formal definition of shear strain: the change in the angle between two initially perpendicular planes. It is measured in radians. However, it is often difficult to take precise measurements of these angular changes, especially for very small deformations. For small deformations, the tangent of the angle ϕ will closely approximate ϕ itself, so that we can approximate the shear strain by

$$\phi \cong \tan\phi = \frac{\delta_s}{L}, \tag{2.5a}$$

so

$$\gamma \cong \frac{\delta_s}{L}. \tag{2.5b}$$

With normal and shear strain defined, we are equipped to address the kinematics of deformation of continuous materials due to loading. We now move on to the second item on our checklist: the internal forces developed in response to external loading.

2.1.3 Measurement of Strain

Until 1930, strain was commonly measured indirectly, using extensometers that measured the displacement ΔL over some initial gage length L to allow strain to be calculated using the equations just discussed. An extensometer system typically included a mechanical or optical lever system. In 1931, the first electrical strain gauge demonstrated that strain could also be measured directly. Most modern strain gauges are resistive electrical meters.

In 1856 Lord Kelvin demonstrated that the resistances of copper and iron wires changed when the wires were stretched, compressed, or otherwise deformed. This concept is at the heart of the electrical strain gauges first implemented by Roy Carlson in 1931 and Edward Simmons in 1938.[2] Advances in materials and fabrication techniques have since refined the design of the resistive strain gauge, whose general construction is shown in Figure 2.6.

When the resistance element (wire grid or metallic foil) is attached to a loaded (and thus deformed) body in such a way that the wire will also be deformed, the measured change in resistance may be calibrated in terms of strain. Important parameters in the design and performance of a strain gauge are (1) the materials used for the wires or foil, and, to a lesser extent, the backing and bonding materials; (2) protection of the gauge; and (3) electrical circuitry, typically involving a Wheatstone bridge. The wires should have a large change in resistance corresponding to the strains expected (sometimes called the wire material's *gauge factor*), a high electrical resistivity, a low temperature sensitivity,[3] and good corrosion resistance, among other factors. Mounting a strain gauge is straightforward (though not always easy) as long as the surface of the body in question is extremely clean and as long as the manufacturer's installation procedures are followed carefully.

2.2 The Method of Sections and Stress

We now want to consider the forces *within* a body that balance the effect of externally applied forces. To do this, we must prepare a free-body diagram (FBD) that shows all the external forces acting on the body at their respective points of application (Figure 2.7a). All of the forces acting on a body, including reactive forces caused by supports and the weight of the body itself (usually not included in a free-body diagram), are considered external forces. This view is valuable but does not allow us to visualize the internal forces

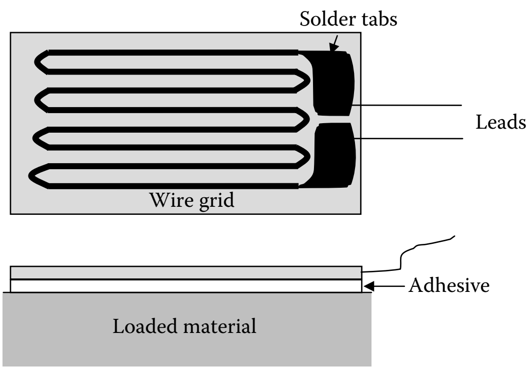


FIGURE 2.6
Construction of a bonded-wire strain gauge.

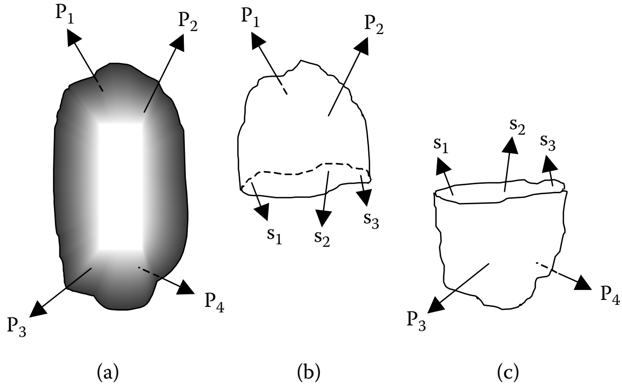


FIGURE 2.7
The method of sections.

we're interested in, so we "slice open" our body (Figure 2.7b and Figure 2.7c). Each sliced section must be in equilibrium, just as the larger body is in equilibrium. The fundamental statement of this is:

> The externally applied forces on one side of an arbitrary cut must be balanced by the internal forces developed at the cut.

The name given to this technique is the *method of sections.*

These internal forces revealed by the method of sections have varying magnitude and direction. They are vectors, and they maintain the externally applied forces in equilibrium. In a solid, these forces determine the solid's resistance to deformations and to external forces.

Physically, these internal forces are what hold the body together: intermolecular forces, or chemical bonds. The application of an external force changes the distance between atoms (i.e., deformation), which changes the forces exerted by these bonds. We could model the internal forces s_i as the resultant of bond forces, but the bookkeeping associated with so many force vectors, and complex atomic arrangements, would be prohibitive. Plus, dealing with continuous materials was supposed to get us off the hook from having to worry about individual atoms, anyway. So, we tend to consider one distributed internal force, and stress is the intensity of that distributed force.

In general, *stress*, represented by sigma, is a force per unit area, or the force's intensity:

$$\sigma = \frac{P}{A}. \tag{2.6}$$

Remember that both the force P and the area A are vectors.[4] The stress depends on the orientations of both P and A, as demonstrated in subsequent chapters. Its units are of force per unit area, generally [N/m²] or [lb/in.²]. It will be useful for us to resolve the internal force P into its components perpendicular and parallel to the section of interest.

Interestingly, it took a long time for engineers and scientists to conceptualize stress as we now understand it. While this was partly due to the susceptibility of scientific progress to fads and biases, and the tyranny of Isaac Newton as a trendsetter (more on this later), it was also a result of researchers focusing on whole structures and not "looking inside" the body as the method of sections demands. Instead, as J. E. Gordon noted, "All through the eighteenth century and well into the nineteenth, very clever men, such as Leonhard Euler and Thomas Young, performed what must appear to the modern engineer to be the most incredible intellectual contortions"[5] to characterize material behavior without the modern notion of stress.

It was Augustin Cauchy who first conceptualized stress and strain as we now understand them, in 1822: "Cauchy perceived that … the 'stress' in a solid is rather like the 'pressure' in a liquid or a gas. It is a measure of how hard the atoms and molecules which make up the material are being pushed together or pulled apart as a result of external forces" (Gordon, 1988, p. 46).

2.2.1 Normal Stresses

By using the method of sections, we can identify the different types of stress. Consider a straight bar acted on at its ends by two equal and opposite forces, as in Figure 2.8a. Remember that these external forces are called tensile forces. Similarly, the bar in Figure 2.9a is acted on by two equal and opposite forces, directed toward each other; these forces are compressive forces. If we make an imaginary cut through each bar and consider the left-hand segment as a free body, as in Figure 2.8b and Figure 2.9b, we see that for each bar to be in equilibrium, a force P_1, equal and opposite to external force P, must exist. This force P_1 is actually an internal force in the original bar that "resists" the action of force P. Also, we assume that the internal resisting force is uniformly distributed over the cross section of the bar. This force per area (the internal force divided by the cross-sectional area) is what we call stress.

The tensile forces of Figure 2.8 produce internal tensile stresses, and the compressive forces of Figure 2.9 produce internal compressive stresses. By convention, tensile stresses are positive, and compressive are negative. (This sign convention has to do with the outward normal vector of surface A, as is discussed in Chapter 3.) Tensile and compressive stresses are developed in a direction perpendicular (normal) to the surfaces on which they act and,

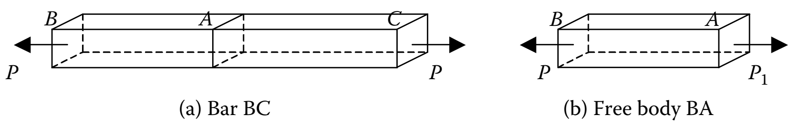


FIGURE 2.8
Bar in tension.

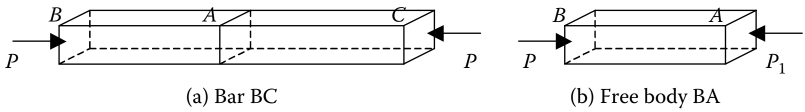


FIGURE 2.9
Bar in compression.

hence, are sometimes called *normal* stresses. We use the Greek letter sigma, σ, to represent normal stress, and we write

$$\sigma \equiv \frac{P}{A}. \tag{2.7}$$

2.2.2 Shear Stresses

Another type of stress, called *shear* stress (sometimes *tangential* stress), is developed in a direction parallel to the surface on which it acts. An example is shown in Figure 2.10. When equal and opposite forces P are applied to two flat plates bonded together by adhesive, the contact (shaded) surface is subjected to a shearing action. In the absence of the adhesive, the two surfaces would slide past one another. The shear force is assumed to be uniformly distributed across the contact area. As a result the shear stress, defined as this shearing force divided by the contact area, is developed. Shear stress can also develop within a single body, when various layers of the material tend to slide with respect to each other.

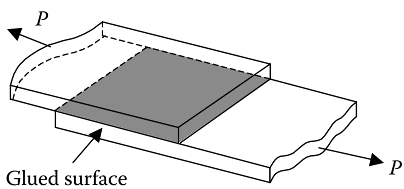


FIGURE 2.10
Shear between two bodies.

Again, stress is the intensity of the internal force and, in this case, is once again P/A, where A is the area of the glued surface; however, for shear stresses, the area A is oriented parallel to the force P, while for normal stresses P is perpendicular to A. (If we more carefully characterize the area A by its outward normal vector, the shear stress is normal to this normal vector, and the normal stress is parallel to it.) We use the Greek letter tau, τ, to represent shear stress:

$$\tau \equiv \frac{P}{A_{\|}}. \tag{2.8}$$

We have included a subscript to remind ourselves that the area A in this expression seems to be parallel to the force P. Now that we have defined both strain (kinematics) and stress, we must consider the relationship between them. We do this in Section 2.3.

Example

Let's consider a structure that might be part of an underwater oil rig turned artificial reef. All members of the truss pictured here have a cross-sectional area of 500 mm², and all the bolts and pin connectors have a diameter of 20 mm. Find (a) the axial or normal stresses in members BC and DE, and (b) the shear stress in the bolt at A if it is in double shear.

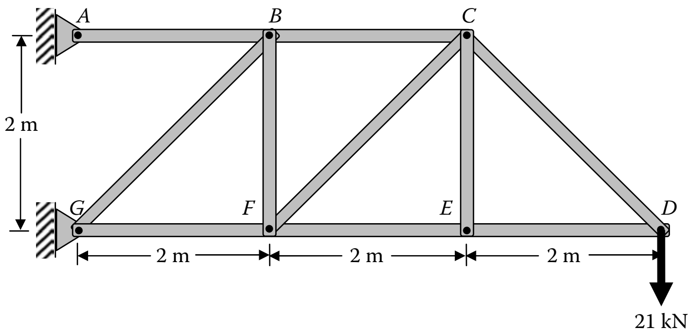


FIGURE 2.11

Given a truss with specified parameters and loading, we must find the requested values of stress. We first examine an FBD of the joint at D:

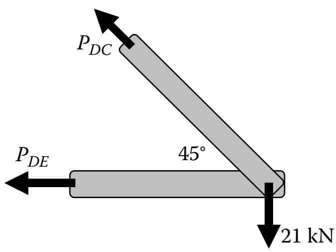


FIGURE 2.12

Note that we have assumed both members DC and DE to be in tension; if we calculate negative values for either internal force, we will know that this assumption was incorrect and that the member is in compression. Since the joint must be in equilibrium we have

$$\Sigma F_y = 0 = P_{DC}\sin 45 - 21\text{ kN} \rightarrow P_{DC} = 29.7\text{ kN}$$

$$\Sigma F_x = 0 = -P_{DE} - N_{DC}\cos 45 \rightarrow P_{DE} = -21\text{ kN.}$$

Using the definition of normal stress we know that

$$\sigma_{DE} = \frac{P_{DE}}{A_{x-\text{sec}}} = -42 \times 10^6 \frac{\text{N}}{\text{m}^2},$$

or

$$\sigma_{DE} = 42 \text{ MPa compressive.}$$

Next, we use the method of sections. We make an imaginary cut between B and C, resulting in an FBD that includes the internal forces in three members of the truss:

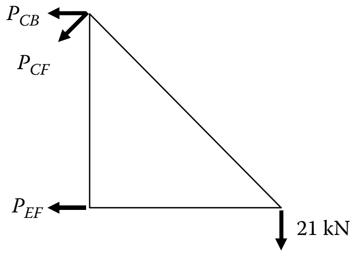


FIGURE 2.13

We apply the third equilibrium equation, summing moments about point F:

$$\Sigma M_F = 0 = P_{CB}\cdot(2 \text{ m}) - 21\text{k N}\cdot (4 \text{ m}) \rightarrow P_{CB} = 42 \text{ kN},$$

so that

$$\sigma_{CB} = \frac{P_{CB}}{A} = 42 \times 10^6 \frac{\text{N}}{\text{m}^2} \quad \sigma_{CB} = 84 \text{ MPa tensile.}$$

We may take this opportunity to check our intuition about this truss. The load, P, is pulling the structure down. Thus, composite member ABC should become longer, and DEFG should become shorter. This would mean that members on the top (like BC) would be in tension and members on the bottom (like DE) in compression. Our results so far are consistent with our physical intuition. This buoys our spirits as we continue to part (b) of the problem, in which we consider the bolt at joint A.

We are told that this bolt is in "double shear." A connection element (bolt or pin) is said to be in "single shear" if one cut between the member and its support is sufficient to break the connection, as shown in Figure 2.14 on the left; "double shear" means that two cuts are needed to break the connection, as on the right. A quick analysis using free-body diagrams of each case should be persuasive evidence that a bolt in double shear experiences half the shear stress of an identically loaded bolt in single shear. This analysis is left as an exercise.

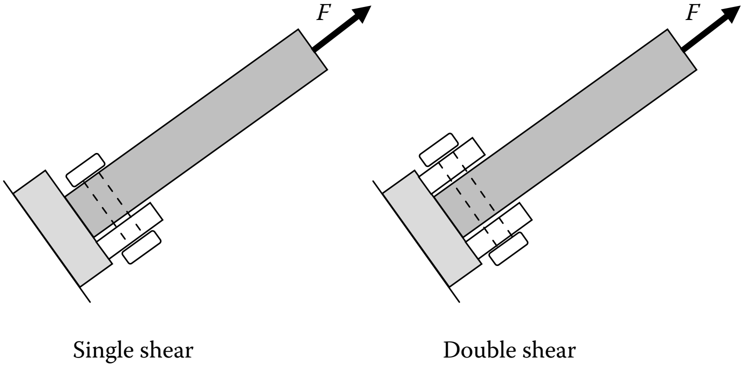


FIGURE 2.14

To find the reaction forces at the supports, we consider an FBD of the entire truss:

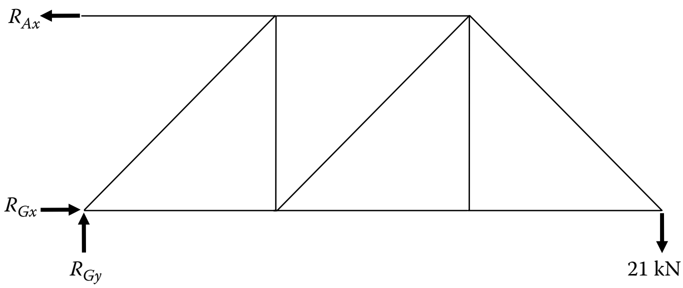


FIGURE 2.15

Summing moments about point G, we have

$$\Sigma M_G = 0 = R_{Ax} \cdot (2\text{ m}) - 21\text{ kN} \cdot (6\text{ m}) \rightarrow R_{Ax} = 63\text{ kN}.$$

So the shear stress in the bolt at A is found:

$$\tau_A = \frac{A_x / 2}{A_{bolt}} = 100 \times 10^6 \frac{\text{N}}{\text{m}^2} = 100\text{ MPa}.$$

2.3 Stress–Strain Relationships

Different materials respond differently to loads. In some materials (e.g., rubber), small loads produce relatively large deformations. Other engineering materials, such as steel, undergo smaller deformations—however, it is still important to consider the effects of such changes. Even very rigid materials, when subjected to a load, will experience a small deformation.

For most engineering materials, a relationship exists between stress and strain. For each increment in stress there is a proportional increase in strain, provided that a certain limit of stress is not exceeded. If the induced stress exceeds the limiting value, the strain will no longer be linearly proportional to the stress. This limiting value is called the *proportional limit*.

Most of the behavior we will consider occurs below the proportional limit, in the regime where stress and strain enjoy a linearly proportional relationship. If we subject a material in this regime to a tensile load P_A, producing a stress σ_A and a strain ε_A, and then subject it to a tensile load P_B, producing stress σ_B and a strain ε_B, and we then plot the stresses and strains, we see a linear relationship between stress and strain, as shown in Figure 2.16.[6]

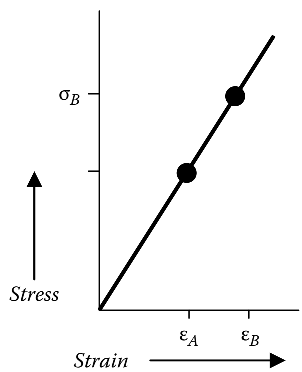


FIGURE 2.16
Linear relationship between stress and strain.

This linear relationship between load and deformation was first stated by Robert Hooke in 1678 and became known as Hooke's law: *Ut tensio, sic vis.* This Latin phrase—in the form of an anagram, *ceiiinosssttuv*—was how Hooke[7] summed up his finding, which he first applied to the extension of a spring. It translates, "As is the extension, so is the force." We have seen his law in this form:

$$F = kx \tag{2.9}$$

and have called k the "spring constant" or "stiffness" of the spring in question. Figure 2.17 shows a representative spring.

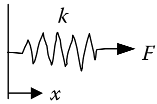


FIGURE 2.17
Linear (Hookean) spring.

The stress–strain diagram is another example of a force (stress being a force per area) being linearly proportional to an extension (strain being extension per initial length). It, too, is Hooke's law: *Ut tensio, sic vis.* It, too, contains a linear constant of proportionality, a *stiffness.*

The ratio of stress to strain, which is also the slope of the line joining these two data points, is constant for loading below the material's proportional limit. This constant is now known as the *modulus of elasticity* or *Young's modulus*, after Thomas Young, who defined it in 1807. (Young's definition was somewhat awkward and ungainly, since Cauchy had yet to clearly define stress. It wasn't until 1826 that Claude Navier defined Young's modulus as we are about to.) The modulus of elasticity for bodies in tension or compression is usually represented by the symbol E and is expressed as

$$E = \frac{stress}{strain} = \frac{\sigma}{\varepsilon}. \tag{2.10}$$

Since strain is a dimensionless quantity (length divided by length), E has the same units as stress: either pounds per square inch (psi) in English units, or N/m^2 or Pascals (Pa) in SI. Table 2.1 shows the values of E for several engineering materials.

Physically, the modulus of elasticity represents the stiffness of a material. A material's stiffness may be defined as the property that enables the material to withstand stress without great strain—in other words, the material's resistance to deformation.

TABLE 2.1
Approximate Design Values (Reflecting Proportional Limits) of Elasticity and Shear Moduli, in Linear Regimes (SI)

Material	Modulus of Elasticity E (MPa)	Modulus of Rigidity G (MPa)
California redwood	7600	
Steel (carbon) ASTM A36	207,000	83,000
Stainless steel	200,000	80,000
Aluminum 6061-T6	70,000	28,000
Glass	48,000–83,000	19,000–35,000
Polycarbonate	2400	800
Concrete	21,500	8970
Bone	1–16,000	4–8000

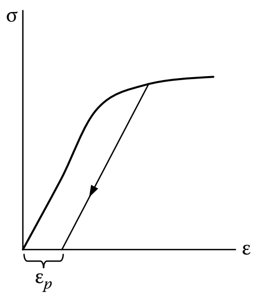


FIGURE 2.18
Plastic deformation incurred when proportional limit is exceeded.

In the Hookean regime, both springs and solid materials are linearly elastic. In the presence of an applied load, stress is linearly related to strain. If an applied load is removed, both stress and strain decrease linearly to zero. However, if a material's proportional limit is exceeded due to an applied load, this is no longer true. In this case, the removal of the applied load causes both stress and strain to decrease linearly, along a line parallel to the linear portion of the stress–strain curve, as shown in Figure 2.18. The strain does not return to zero. By exceeding its proportional limit, the material has undergone a permanent *plastic* deformation. Plastic, as opposed to elastic, deformation represents a permanent set of the material. For most materials, the degree of plastic deformation depends on both the maximum stress value reached and the time elapsed before the load is removed. The stress-dependent portion of plastic deformation is known as *slip*, and the time-dependent part, which can also be influenced by temperature, is known as *creep*.

Shear stress is also proportional to shear strain, as long as the stress is below the proportional limit. The constant of shear proportionality is known as the *shear modulus* or the *modulus of rigidity*. It is represented by G and expressed as

$$G = \frac{\textit{shear stress}}{\textit{shear strain}} = \frac{\tau}{\gamma}. \tag{2.11}$$

Average values of the modulus of rigidity for some common materials are given in Table 2.1. Note that the moduli of elasticity and rigidity differ significantly for each material.

It is interesting to observe the consistency of the ratio of E to G, despite the diversity of materials represented in Table 2.1. In Section 3.1 we reflect further on the relationship between E and G, representing a material's resistance to axial deformation relative to its resistance to shear.

We now have two additional forms of Hooke's law, likenesses of $F = kx$ for one-dimensional loading. We see this likeness clearly by rearranging the two equations:

$$\sigma = E\varepsilon. \tag{2.10}$$

$$\tau = G\gamma. \tag{2.11}$$

In modeling our material body as a linear spring, we are making the assumption of linearity (small deformations, i.e., that we are in the Hookean regime of the material's stress–strain curve). This model incorporates three further assumptions that thus represent limitations—albeit broad ones—on the kinds of materials it can represent. One assumption is that the material is *homogeneous*, by which we mean the material constants (e.g., Young's modulus) do not vary from point to point—that is, are not functions of the coordinates. The second assumption is that the material is *isotropic*, by which we mean that the elastic properties are invariant with respect to any rotation of the coordinate axes. In other words, no matter which axis we look down, we see the same material behavior. The third assumption is that there is no apparent effect of temperature in our simple version of Hooke's law. We incorporate the effects of temperature in Section 2.9.

Each material has its own characteristic stress–strain curve. The extreme values of strain that materials can withstand vary widely, as do the slopes of the Hookean portions of their curves, as shown in Figure 2.19. The terminal point on a stress–strain diagram represents the complete failure (rupture or fracture) of the specimen. Materials that are capable of withstanding large strains without a significant increase in stress (and that may be thought of as "stretchy") are called *ductile* materials. Low-carbon steels, polymers, skin, and rubber are examples of ductile materials. *Brittle* materials, on the other hand, will experience a huge increase in stress from even a small strain and will fail abruptly after a small amount of deformation. Cast iron, glass, ceramics, concrete, and bone are examples of brittle materials. Further discussion of material properties is available in Section 3.6 and Section 3.7.

For the most part, we consider homogeneous, isotropic materials—materials whose behavior does not depend on the direction (e.g., tension or compression) of loading. Many engineering materials such as metals and ceramics may be readily modeled this way; however, some materials, like wood and bone, have different properties in different directions. Wood is strongest against loading along its grain and is much easier to break with loads applied across the grain; compact bone is strongest along its long axis to resist compressive loading. For the time being, we neglect such variations and cling to the assumptions of homogeneity and isotropy.

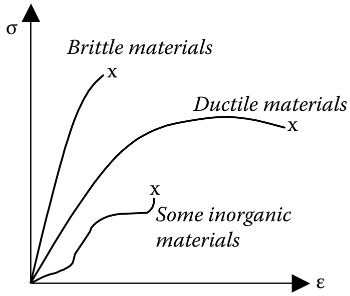


FIGURE 2.19
Schematic of typical stress–strain diagrams. See Sections 3.6 and 3.7 for further discussion of the terms *ductile* and *brittle*.

We recall our checklist of what is needed to apply continuum mechanics to understand the response of a body to external loading: We must (1) characterize the deformation of a continuous material, (2) define the internal loading and (3) relate this to the body's deformation, and (4) make sure that the body is in equilibrium. We have accomplished the first three items on the list and now understand that in doing so we have constructed (1) a kinematic description of deformation, or strain; (2) a definition of stress; and (3) a constitutive law relating stress and strain. The last item on our list, (4) equilibrium, is addressed by the method of sections; we also consider equilibrium more rigorously in the following section.

2.4 Equilibrium

We have used equilibrium and the method of sections to apply Newton's second law on a "macroscopic" basis. Now we will do a "microscopic" equilibrium analysis in terms of the stress resultants at an arbitrary point in the bar, acting on an infinitesimal element of length dx and of volume $d\mathcal{V} = A(x)dx$, as shown in Figure 2.20. Since the point we have chosen is arbitrary, this analysis is valid at every point in the bar—and so for the entire bar.

Summing forces in the x direction on this uniaxially loaded element, we see that the *internal* axial force N balances both the *external* axial load $q(x)$, a *distributed* axial load per unit length of the bar (a force per length, having units of N/m or lb/ft), and an axial *body force*, B_x (a force per volume):

$$\left(N(x)+\frac{dN(x)}{dx}dx\right)-N(x)+q(x)dx+B_xA(x)dx=0. \tag{2.12}$$

The internally distributed body force allows us to include forces that depend on intrinsic mass or volume, such as gravity or magnetic fields. For example, to consider the weight of a vertical element, we would use $B_x = \rho g$ if x points toward the center of the Earth. Equation (2.12) can then be simpli-

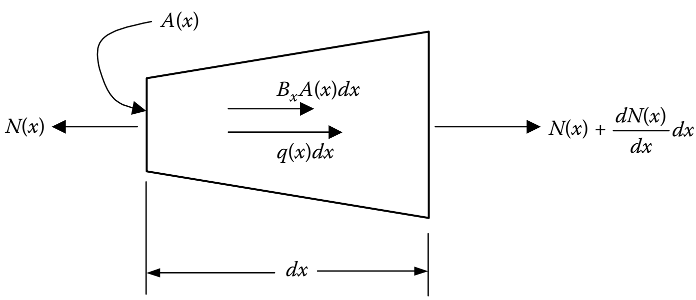


FIGURE 2.20
Equilibrium of an infinitesimal element in one dimension: Internal axial force N balances applied axial load $q(x)$, and body force B_x.

fied, yielding an ordinary differential equation of first order for the axial normal stress resultant:

$$\frac{dN(x)}{dx} + q(x) + B_x A(x) = 0. \tag{2.13}$$

The axial loads are generally "distributed" as concentrated loads P_i located at coordinates x_i, in which case

$$q(x) = \sum_i P_i \delta(x - x_i),$$

and Equation (2.13) takes the form

$$\frac{dN(x)}{dx} + \sum_i P_i \delta(x - x_i) + B_x A(x) = 0. \tag{2.14}$$

Section 2.7 shows that microscopic and macroscopic equilibrium results are in agreement. Our checklist for continuum mechanics analysis is complete:

- ✓ Kinematics (strain)
- ✓ Definition of stress
- ✓ Constitutive law (stress–strain relationship)
- ✓ Equilibrium

Now that we have developed these four items for one-dimensional loading, we will see what they mean for an axially loaded bar like those in our underwater structure.

2.5 Stress in Axially Loaded Bars

Consider a steel ruler—a thin body made of a seemingly compliant material. We know that if we hold such a ruler by one end and push down on the other end (perpendicular to the ruler's broad surface), as in Figure 2.21a, the loaded end will be deflected significantly. In this case of loading, we call the system a cantilever *beam*. On the other hand, if we instead pull on the free end (parallel with the long axis of the ruler), as shown in Figure 2.21b, we would see very little movement. A system with this type of loading is called a *bar*. It is intriguing that the same body can experience such dramatically different behavior due to differences in loading. We hope to be able to postulate and develop models to explain these different behaviors.

Once we remove either load from the ruler—once we stop pushing or pulling—the ruler returns to its original, planar shape. In this way, the ruler behaves like an elastic spring, just as Hooke suggested. In our "beam" and "bar" experiments, the different behavior of the ruler can be explained by its having a different stiffness depending on the loading. Later in this text, we derive the different forms of this stiffness and see in detail that the beam stiffness is much less than the stiffness in the bar mode, which is why we see greater movement or deflection when the ruler acts like a beam.

For now, the important lesson is that the effective stiffness (a measure of how much a body will resist being deflected by a load) of a structural element or mechanical device is dependent on several factors, including the nature of the loading, as well as the element's geometry and the material itself. Since we are interested in how bodies will react to external forces, this stiffness provides us with a way to quantify their reactions.

Let's expand our ruler example of a bar in axial loading (Figure 2.22a). The bar is built in, or attached to a wall, at $x = 0$ and is subjected to a single external (applied) load P at $x = L$. The load P acts along the bar's axis. We know from Newton's second law that to keep the bar in static equilibrium, the attachment point or wall must exert an equal and opposite force P at the left end of the bar.

What we're interested in, of course, is what's happening inside the bar. We can use the method of sections to make an imaginary slice along the bar, exposing a cross section of area A. A free-body diagram will show us that something must be happening on that area to exert a net tensile force P across A. And, if our slice is normal to the bar's axis (as in Figure 2.22b), the exposed area A is also normal to the axis, and we can define the normal stress, σ, acting on that area as we did in Equation (2.7):

$$\sigma \equiv \frac{P}{A}. \tag{2.7}$$

If instead we make our section cut at an angle, θ, the picture will be different (Figure 2.22c). Now, the equilibrating force at the section surface has two

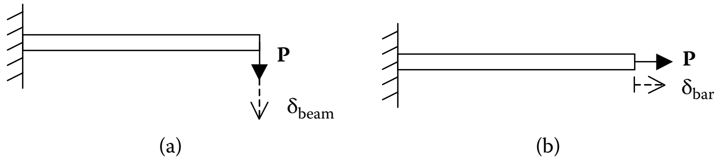


FIGURE 2.21
Illustration of beam and bar modes.

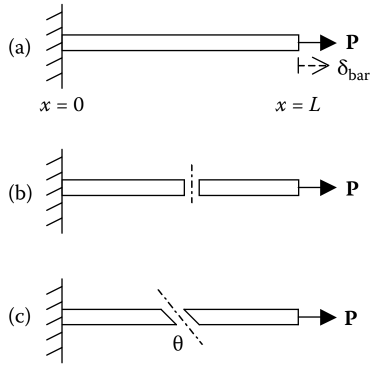


FIGURE 2.22
Stresses on axially loaded bar.

components, as shown in Figure 2.18. The normal force component is $P\cos\theta$, and the shear component (parallel to the section surface) is $P\sin\theta$. (These components may be obtained by summing forces in the x and z directions.)

The area of the inclined cross section is $A/\cos\theta$. From these values we can calculate the normal stress σ_θ and the shear stress τ_θ on this angled section by the two equations:

$$\sigma_\theta = \frac{force}{area} = \frac{P\cos\theta}{A/\cos\theta} = \frac{P}{A}\cos^2\theta, \tag{2.15a}$$

$$\tau_\theta = -\frac{P\sin\theta}{A/\cos\theta} = -\frac{P}{A}\sin\theta\cos\theta. \tag{2.15b}$$

The negative sign in the equation for tau comes about due to the sign convention for shear stresses (the shear force $P\sin\theta$ is in the negative y' direction).

Both normal and shear stresses, we have seen, will vary with the angle θ. Looking at equation (2.15a) and equation (2.15b) for σ_θ, we see that it will reach its maximum value when $\theta = 0^\circ$, that is, when the section is perpendicular to the axis of the bar (as in Figure 2.22b). The corresponding shear stress at $\theta = 0^\circ$ would be zero. Hence we determine the maximum normal stress in an axially loaded bar:

$$\sigma_{\max} = \frac{P}{A}. \tag{2.16}$$

A question to think about is: what happens at $\theta = 90^\circ$? Does this make sense?

If we differentiate the equation for shear stress with respect to angle θ and set it equal to zero, we should find the maximum value of $\tau\theta$. We find that $\tau\theta$ has its maximum value when $\tan\theta = \pm 1$, leading us to the conclusion that τ_{max} occurs on planes of either +45° or –45° with the bar axis. If we substitute ±45° into our equation, we find that

$$\left|\tau_{\max}\right| = \frac{P}{2A} = \frac{\sigma_{\max}}{2}. \tag{2.17}$$

Thus, the maximum shear stress in an axially loaded bar is only half as large as the maximum normal stress.

To consider the stresses on the section formed by a "cut" at the angle $\theta - 90°$, a section perpendicular to the θ section, we can either examine the figure on the right side of Figure 2.23, or substitute $\theta - 90°$ in for θ in the equations we have for σ_θ and τ_θ. Either way, we will find that

$$\sigma_{\theta-90^\circ} = \frac{force}{area} = \frac{P\sin\theta}{A/\sin\theta} = \frac{P}{A}\sin^2\theta, \tag{2.18a}$$

$$\tau_{\theta-90^\circ} = \frac{P\cos\theta}{A/\sin\theta} = \frac{P}{A}\sin\theta\cos\theta. \tag{2.18b}$$

2.6 Deformation of Axially Loaded Bars

We've established expressions for stress, strain, and the modulus of elasticity E. These may now be combined into a convenient expression to directly determine the total deformation δ for an axially loaded bar (Figure 2.22a). We begin with the definition of modulus of elasticity, or Hooke's law, and substitute for stress and strain:

$$E = \frac{\sigma}{\varepsilon} = \frac{P/A}{\delta/L} = \frac{PL}{A\delta}. \tag{2.19}$$

Then, solving for δ, we obtain

$$\delta = \frac{PL}{AE}, \tag{2.20}$$

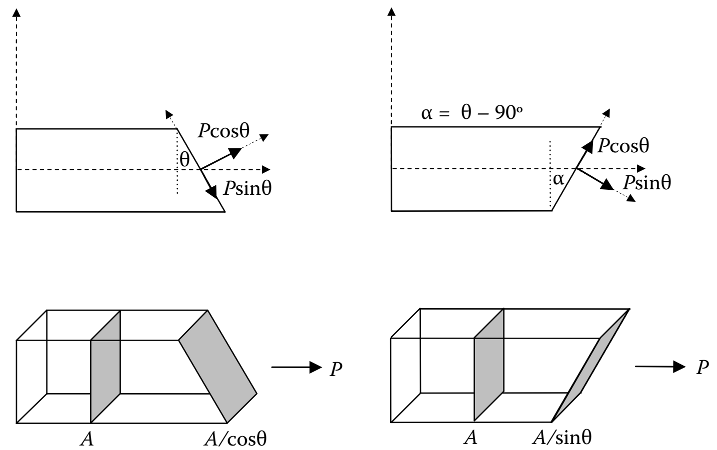


FIGURE 2.23
Sectioning of a bar at angle θ (left) and angle $\theta - 90°$(right) from vertical.

where

δ = the total axial deformation, with dimensions of Length [(in.), (m, mm)]

P = the total applied external axial load, with dimensions of Force [(lb, kips), (N)]

L = the original length of the bar, with dimensions of Length [(in.), (m, mm)]

A = the cross-sectional area of the member, with dimensions Length^2 [(in.^{2}), (m^2, mm^2)]

E = the modulus of elasticity with dimensions of Force/Length^2 [(psi, ksi), (Pa, MPa)]

This expression is valid only when the stress in the bar does not exceed the proportional limit. This should make sense, as it is only below this limit that the bar's stress and strain will obey Hooke's law. Also, equation (2.20) assumes that the forces, area, and properties of the bar do not change along its length. For a more complex problem, where quantities vary along the bar's axis (here the x axis), we can obtain a similar relationship that takes such variations into account:

$$\delta = \int_0^L \frac{P(x)}{A(x)E(x)} dx . \tag{2.21}$$

We can cast this relationship in terms of the bar's stiffness, as discussed earlier. If we recall the form Hooke's law took for linear springs, $F = kx$, we can write P as a function of δ using equation (2.20):

$$P = \frac{AE}{L}\delta . \tag{2.22}$$

Comparing equation (2.22) with $F = kx$, we see that the axial deformation δ of this bar due to the axial load P depends on its stiffness, AE/L. Chapter 6 compares this spring stiffness with that for a beam, loaded as in Figure 2.16.

2.7 Equilibrium of an Axially Loaded Bar

Now we want to combine our kinematics (equation 2.4) and constitutive (equation 2.10) and equilibrium (equation 2.14) equations to characterize a uniaxially loaded bar. In principle, this is a system of three equations for three unknowns: the strain ε, the stress σ, and the axial displacement $u(x)$. However, we can simplify the mathematics by eliminating variables and reducing our system to a single differential equation. Since our system of equations includes two first-order differential equations (equilibrium, kinematics) and one algebraic equation (Hooke's law, our constitutive equation), we expect our single equation to be second order. We achieve this result by, first, writing the stress in terms of strain and strain in terms of the displacement, $u(x)$, that is,

$$\sigma = E\varepsilon = E\left(\frac{du(x)}{dx}\right). \tag{2.23}$$

Second, we substitute equation (2.23) into the equilibrium equation (2.14) to find (assuming that the area, the elastic modulus, and the temperature change are all constant—that is, they do not vary with the x coordinate)

$$E\frac{d^2u(x)}{dx^2} + B_x = 0 . \tag{2.24}$$

This is the second-order equation we expected. In the absence of body forces ($B_x = 0$) it is easily integrated, yielding

$$u(x) = C_1 x + C_2 . \tag{2.25}$$

To determine the constants of integration in equation (2.25), we must apply appropriate boundary conditions. As an example, we solve for the diplacement in the bar shown in Figure 2.22a. One boundary condition is clear: The displacement (or movement) of the bar is zero at the left end ($u(0) = 0$) because the bar is attached to the wall and restrained there. At the "free" end, $x = L$, we are pulling with a force P so that we can express this boundary condition in terms of the strain as

$$\frac{du(L)}{dx} = \varepsilon(L) = \frac{\sigma}{E} = \frac{P}{AE}. \tag{2.26}$$

After applying our two boundary conditions, we find the solution (2.25) to be

$$u(x) = u(0) + \frac{Px}{AE} = \frac{Px}{AE}. \tag{2.27}$$

The net extension of an entire bar (or rod) of length L is thus

$$\delta = u(L) - u(0) = \frac{PL}{AE}, \tag{2.28}$$

which is in agreement with equation (2.20) and from which we can recover the expression for the bar stiffness, AE/L.

2.8 Indeterminate Bars

For some structural systems, the equations for static equilibrium expressed in terms of stresses[8] are insufficient for determining reactions. This may be because some of the reactions are superfluous or redundant for maintaining equilibrium. But even a redundant support feels reaction forces—forces we as engineers must calculate. Equilibrium equations may also be insufficient when some internal forces cannot be determined using the equations of statics alone. Both of these situations, called *statical indeterminacy,* may arise in axially loaded bar systems.

We can resolve statical indeterminacy by several methods. In all of the available methods, as in all of our mechanics problems, we must make sure of three things, in no prescribed order:

- Equilibrium conditions for the system must be assured, both locally and globally.
- Geometric compatibility must be satisfied among deformed parts of the body and at boundaries. This has to do with the kinematics of deformation.
- Constitutive relations such as Hooke's law must be obeyed by all materials of the system.

Of the available methods, the two most commonly used are (1) the *force* method, in which we first remove and then restore a redundant reaction;

and (2) the *displacement* method, in which we maintain compatibility of the displacements of adjoining members and at the boundaries and in which solution displacements are obtained from equilibrium equations. The displacement method is the basis for most of the finite element method (FEM) programs that are commonly used to analyze complex structures and is better suited to large systems. Both methods make use of the analogy between Young's modulus E and our old friend the spring constant k. E and k each relate force and displacement in a linear equation: $\sigma = E\varepsilon$ and $F = kx$.

We have just seen that the stiffness of an axially loaded bar may be expressed as $k = AE/L$.

2.8.1 Force (Flexibility) Method

The force method is also sometimes called the force/flexibility method. We will be thinking of our indeterminate bars as elastic members of a system, each bar with a *flexibility* f related to its stiffness k. In fact, f is defined as the reciprocal of k:

$$f = 1/k = \Delta/P,$$

or

$$L/AE.$$

Note that f has physical dimensions of displacement/force, reciprocal dimensions of the stiffness k.

To illustrate the force method, consider the following example. In Figure 2.24a, an axial force P is applied at point B of the varying-diameter bar ABC. This axial load leads to reactions R_1 and R_2 being developed at both ends, and the system deforms to the state seen in Figure 2.24b. The deformations shown are exaggerated.

Since only one nontrivial equation of statics is available ($\Sigma F_x = 0$, with two unknowns R_i), this system is statically indeterminate to the first degree. We will assume positive forces and deflections so that any result with a negative sign will mean that the force or deflection in question is in the opposite direction from that drawn in Figure 2.24b. The force method tells us to "remove" one of the reactions (in the same hypothetical sense that we "slice" bodies open to use the method of sections). We choose to remove the right-hand reaction R_2 first. This permits the system to deform, as in Figure 2.24c.

We see that in Figure 2.24c, the same axial deformation Δ_o occurs at B as at C—in the imagined absence of reaction R_2 (imposed by the right wall), the bar is free to deform in this way. If the flexibility of the narrower elastic bar is f_2, we can use the definition of flexibility to write

$$\Delta_o = f_2 P. \tag{2.29}$$

But this deformation violates the geometric condition that is actually imposed at A: There is, truly, a wall that prevents a deflection of even Δ_o. To comply with geometric compatibility, we must find the deflection Δ_1 that would be caused by R_2 on the unloaded bar, as shown in Figure 2.24d. This deflection is caused by the stretching (if R_2 is in the direction shown; otherwise, the compression) of both constituent bars. Thus,

$$\Delta_2 = \frac{R_2 L_1}{A_1 E_1} + \frac{R_2 L_2}{A_2 E_2} = (f_1 + f_2)\, R_2. \tag{2.30}$$

We may then achieve compatibility by requiring that

$$\Delta_o + \Delta_1 = 0. \tag{2.31}$$

That is, there is no net deformation of the actual bar system. From this expression we find an expression for R_2:

$$R_2 = -\frac{f_2}{f_1 + f_2} P. \tag{2.32}$$

The negative sign here indicates that R_2 acts in the opposite direction from what we'd assumed: The bar is in compression. (The same is true for its deflection, Δ_1. It is negative, reflecting the fact that the bar is being compressed.)

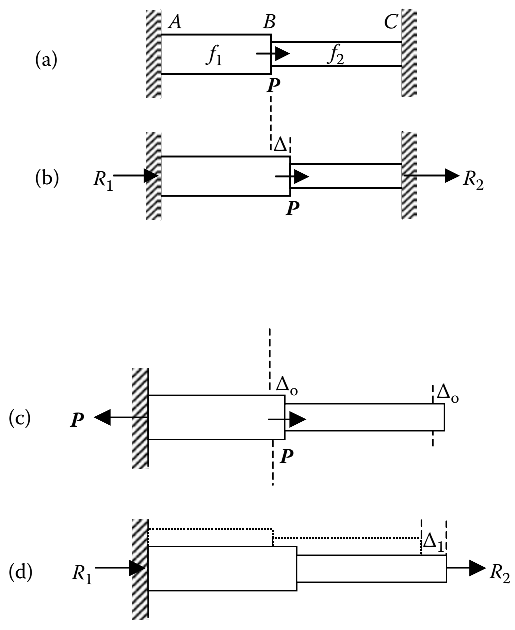


FIGURE 2.24
Decomposition of indeterminate bar by force method.

The idea of the force method is that the complete solution is the sum of the solutions shown in Figure 2.24c and Figure 2.24d; the method is an application of the principle of *superposition*. Our premise is that the resultant stress or strain in a system due to several forces is the algebraic sum of these forces' individual effects. This is only true if each effect is linearly related to the force causing it—that is, if we are in the Hookean range of behavior.

It may be useful to refer to the steps of the force method in problem solving:

1. Determine the number of redundants—that is, the number of forces that cannot be determined from equilibrium alone. The number of forces needed to maintain equilibrium is equal to the number of equations of equilibrium, so any additional forces are "redundant."
2. Choose some of the reactions to be the redundants and remove them from the structure, thus temporarily producing a determinate structure. There is no formal method or set of criteria for making the choice, so convenience, as viewed through the lens of experience, is the guiding principle for choosing redundants.
3. Calculate the displacements at the points from which redundants were removed, as produced by the actual (given) external loading.
4. Calculate the displacements at the points from which redundants were removed but now as produced by the redundants without the given external loading.
5. Sum the two displacements at each point where a redundant has been removed, as calculated in the last two steps—that is, as displacement (step 3) + displacement (step 4). Applying superposition to this linear structure, we see that we must add the actual displacement at that point of the fully loaded, indeterminate structure. We then calculate the values for the redundant forces from these equations. (We are enforcing *compatibility* or *consistency of deformations* when we perform this step.)
6. With the redundants determined in step 5 acting, determine the remaining support reactions of the fully loaded, indeterminate structure by applying equilibrium.

This procedure is very general; in practice, any number of axial loads, bar cross sections, material properties, and thermal effects on the length of a bar system may be included in your analyses. However, for very large systems, application of the force (flexibility) method is very difficult.

2.8.2 Displacement (Stiffness) Method

The displacement method is also known as the stiffness method. We remember that the stiffness of an axially loaded bar may be expressed as

$$k = AE/L.$$

If we are presented with a statically indeterminate elastic axially loaded bar system (like that in Figure 2.25a), we may define the stiffness of each member k_i as

$$K_i = A_i E_i / L_i.$$

An applied force P at point B causes reactions R_1 and R_2. As before, these forces and the displacement Δ at B are considered positive when they act toward the right.

Our objective is to determine the displacement Δ. (Since there is only one unknown Δ to be determined in this example, this problem is said to have one degree of kinematic indeterminacy, or one degree of freedom.) We also hope to find expressions for the reaction forces R_i.

In the problem considered here (Figure 2.25), the displacement Δ at B causes tension in bar AB and compression in bar BC. Because we understand this, we can assume the senses of the reaction forces as shown in Figure 2.25b. So, if k_1 and k_2 are the stiffnesses of the two bars, the respective internal forces are $k_1\Delta$ and $k_2\Delta$. These internal forces and reactions are shown on isolated free bodies at points A, B, and C in Figure 2.25c. These points are called *nodes*, or *nodal points*. The sense of the internal forces is known, since AB is in tension and BC is in compression. Writing an equilibrium equation for the free body at node B, we have

$$-k_1\Delta - k_2\Delta + P = 0, \tag{2.33a}$$

$$\Delta = \frac{P}{k_1 + k_2} b. \tag{2.33b}$$

Equilibrium for free bodies A and C gives us

$$R_1 = k_1\Delta \text{ and } R_2 = k_2\Delta. \tag{2.34}$$

So, synthesizing these three results, we find that

$$R_1 = \frac{k_1}{k_1 + k_2} P \text{ and } R_2 = \frac{k_2}{k_1 + k_2} P \tag{2.35}$$

in the directions indicated in Figure 2.25b, such that AB is in tension and BC in compression.

It may be useful to refer to this sequence of steps for the displacement method:

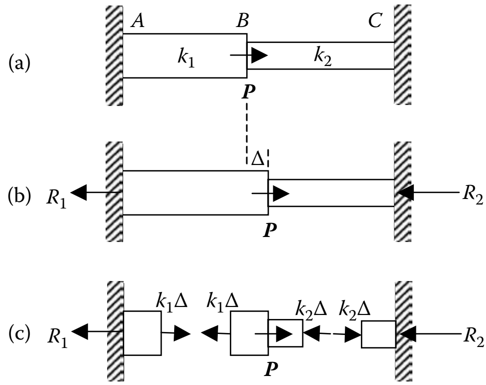


FIGURE 2.25
Displacement method for statically indeterminate bar.

1. Determine the number of redundants—that is, the number of forces that cannot be determined from equilibrium alone.
2. Identify within the structure a number of points equal to the number of redundants, and for each of these points identify a nodal displacement of the structure.
3. Calculate the forces needed to "produce" the nodal displacement, and sum all the forces at the nodes to enforce equilibrium.
4. Eliminate the nodal displacements from the nodal equilibrium equations to calculate the unknown nodal forces.
5. Determine the support reactions of the fully loaded, indeterminate structure by applying equilibrium.

2.9 Thermal Effects

So far, we have considered mechanical stress and externally applied loads as the only sources of strain in materials. With changes of temperature, however, solid bodies expand with increasing temperature and contract with decreasing temperature. These deformations produce *thermal strains*. We define thermal strain ε_T in the following way:

$$\varepsilon_T = \alpha\,(T - T_o) = \alpha\,\Delta T, \tag{2.36}$$

where α is an experimentally determined coefficient of (linear) thermal expansion, and T_o and T are the initial and final temperatures of our material of interest. The thermal expansion coefficient α measures dimensional change per degree of temperature change for a given material. Typical values in SI units of (m/m)/°C, or just $(°C)^{-1}$, range from 9.9×10^{-6} for concrete to 11.7×10^{-6} for carbon steel to 23×10^{-6} for aluminum.

Thermal strain has no directional dependence; equal thermal strains develop in every direction for unconstrained homogeneous isotropic materials. For a body of length L subjected to a temperature change, the extensional deformation δ_T is

$$\delta_T = \alpha\,(\Delta T)\,L, \tag{2.37}$$

where ΔT is allowed to be positive or negative for increasing or decreasing temperature.

If the body in question is free to expand or contract (i.e., the body is not restrained), no stress is induced by these thermal effects. The dimensional change δ_T will simply occur, and the otherwise unloaded bar will continue to be in equilibrium. However, if the body is partially or fully restrained so as to prevent this change δ_T, internal thermal stresses will develop. Thermal stress for a temperature change ΔT is given as

$$\sigma_\mathrm{T} = E\,\alpha\,(\Delta T). \tag{2.38}$$

If this body is fully restrained and then cooled, the stress induced is tensile; if the body is fully restrained and then heated, the stress induced is compressive. The stresses and strains due to thermal effects may be combined with the stresses and strains in the same directions by straightforward superposition.

2.10 Saint-Venant's Principle and Stress Concentrations

In applying equations such as $\sigma = P/A$, we have assumed that forces and stresses are distributed uniformly across their surfaces of action. In ideal cases such as the axially loaded bars of the previous sections, this is very nearly the true situation. However, in more realistic scenarios, things are more complex. Fortunately for us, many researchers have performed detailed calculations of stress states, and have learned things from the distributions they found. We may benefit from their conclusions without performing arduous computations ourselves.

An exemplary such result came from the analysis of an elastic block, acted on by concentrated forces at its ends, as in Figure 2.26a. (Of course, in the real world, a truly concentrated force such as this one is not even possible.) The calculated stress distributions at three incremental depths within the bar are shown in Figure 2.26b, Figure 2.26c, and Figure 2.26d. Clearly, these are not uniform distributions across the cross section.

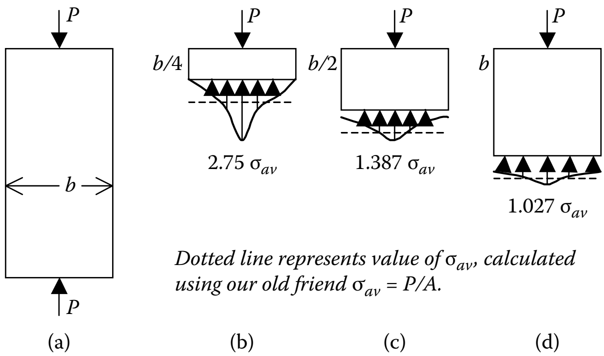


FIGURE 2.26
Stress distribution near concentrated force in plate.

Two important facts may be gleaned from these results. One, the average stresses as calculated by our formulas (stress = force/area) are in agreement with these more carefully obtained numbers. Two, the normal stresses are nearly uniform on a surface whose distance from the applied force is the same as the width of the body. (This is true despite the high spatial variation in stress at surfaces nearer to the force application.) This second point illustrates *Saint-Venant's Principle,* as first stated by the eponymous French elastician in 1855. It means that the manner of force application (point, or evenly distributed, or other) has a significant effect on the stress distribution only in the near vicinity of the force's application. We are applying this principle when we idealize our systems.

Highlighted in Figure 2.26b, Figure 2.26c, and Figure 2.26d are the maximum normal stresses at each cut and their proportionality to the average stress. This maximum stress and its relation to average stress is a function of geometry. In particular, features such as holes and filleted edges cause areas of stress concentration and ruin our idealization of uniform stress distribution. A formula is available for the calculation of maximum normal stress:

$$\sigma_{\max} = K\sigma_{av} = K\frac{P}{A}, \tag{2.39}$$

where K is an experimentally obtained stress concentration factor for the particular geometric feature in question. Figure 2.27 shows stress concentration factors for flat axially loaded members with three types of change in cross section.

Whether the area A used in equation (2.39) is the original area (without a hole) or the reduced area can vary with researcher and data; this naturally affects the value of K. The data in Figure 2.27 are based on the reduced cross section. In cases not covered by the graph in Figure 2.27, another reference (e.g., *Peterson's Stress Concentration Factors*, by Walter Pilkey (1997)) or an online stress concentration calculator may prove useful.

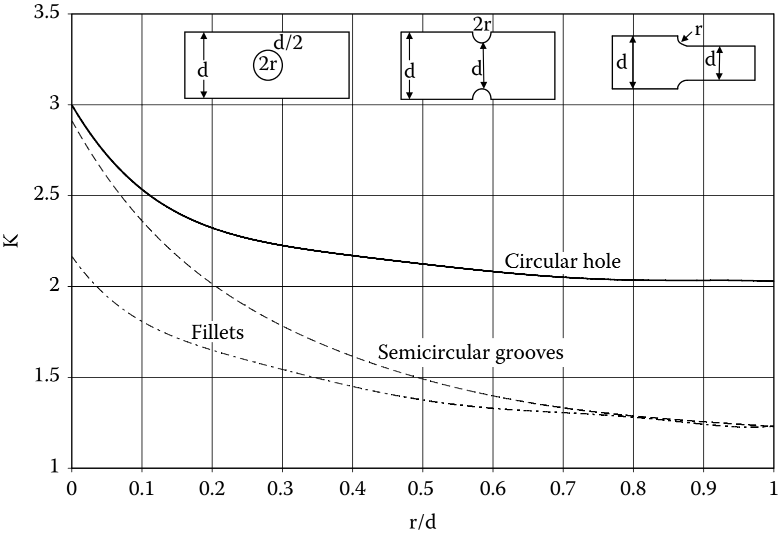


FIGURE 2.27
Stress concentration factors for flat bars. (Adapted from M. M. Frocht, *ASME Journal of Applied Mechanics* 2, A67–A68, 1935.)

In ductile materials, high stress concentration is not necessarily dangerous because these materials can accommodate high stresses through plastic yielding and subsequent stress redistribution. In brittle materials, cracks may occur in areas of high localized stress.

2.11 Strain Energy in One Dimension

Thanks to Robert Hooke, we have recognized that a solid material responds to loading in much the same way as a linear spring, as long as the material remains below its proportional limit. Recall that the linear elastic spring is an energy storage device for which we can calculate the stored energy as

$$U_{spring} = \int_0^x F_s dx = \int_0^x kx dx = \frac{1}{2} kx^2. \tag{2.40}$$

We can also calculate the strain energy stored in a deformed elastic solid. For the elementary one-dimensional Hooke's law, the *strain energy density,* U_0, or strain energy per unit volume (check the dimensions!) can be calculated as the work done by a stress state acting through its corresponding strain:

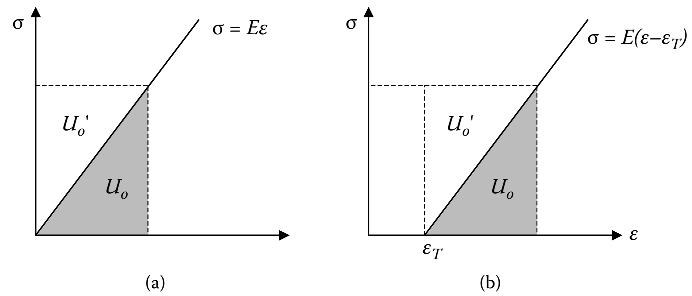


FIGURE 2.28
Stress-strain relationship and energy densities (a) without and (b) with internal stresses.

$$U_0 = \int_0^{\varepsilon} \sigma d\varepsilon = \int_0^{\varepsilon} E\varepsilon d\varepsilon = \frac{1}{2} E\varepsilon^2. \tag{2.41}$$

As with the comparable spring calculation, we recognize U_0 as the area below the stress–strain curve given by Hooke's law, as is shown in Figure 2.28a. The area above the stress–strain curve is the *complementary energy density*:

$$U_o^C = \frac{\sigma^2}{2E}. \tag{2.42}$$

The important point to note here is that while the strain and complementary energy densities are obviously equal, we refer to the strain energy density when the expression is cast in terms of strains or displacements, and we refer to the complementary energy density when the corresponding expression is written in terms of stresses or forces. For the spring the comparable formulas would be

$$U_{spring} = \frac{1}{2} kx^2, \tag{2.43a}$$

$$U_{spring}^C = \frac{F_s^2}{2k}. \tag{2.43b}$$

When thermal stresses are included, the calculation is somewhat less straightforward because of the offset of thermal strain along the strain axis, as shown in Figure 2.28b. Thus, here the strain energy density integration would take the form

$$U_0 = \int_{\varepsilon_T}^{\varepsilon} \sigma d\varepsilon = \int_{\varepsilon_T}^{\varepsilon} E(\varepsilon - \varepsilon_T) d\varepsilon = \frac{1}{2} E(\varepsilon - \varepsilon_T)^2, \tag{2.44}$$

which in expanded form can also be written as

$$U_0 = \frac{1}{2} E\varepsilon^2 - E\varepsilon(\alpha T) + \frac{1}{2} E(\alpha T)^2 . \tag{2.45}$$

As a final note, which is easily verified, the strain and complementary energies must always satisfy the requirements (and produce the results) that

$$\frac{\partial U_o(\varepsilon)}{\partial \varepsilon} = E\varepsilon \equiv \sigma, \tag{2.46a}$$

$$\frac{\partial U_o'^C(\sigma)}{\partial \sigma} = \frac{\sigma}{E} \equiv \varepsilon. \tag{2.46b}$$

2.12 A Road Map for Strength of Materials

For one-dimensional loading, we have addressed the checklist for continuum mechanics, involving (1) kinematics, or description of deformation; (2) a definition of stress; (3) a relationship between stress and strain; and (4) equilibrium. We must next turn our attention to loading in multiple dimensions so that we may model more realistic problems. If we look back at our modeling of stretched or compressed bars, we can discern a pattern of thought that serves as a road map for a more general approach to problems in strength of materials, structural analysis, and elasticity.

Our road map encompasses six major physical elements, beginning with the external loads—the given, applied loads on a solid. These loads or forces are the "drivers" of our analyses because, as engineers, we design structures and machine elements to support, guide, and contain the effects of the external loads. This was illustrated in our analysis of a long, thin bar that was being pulled (or pushed) by an axial force.

The reactions are external forces that support the loaded body and keep it from moving in response the given applied loads. They are determined by requiring the body in its entirety to be in equilibrium under the given externally applied loads. There are many kinds of reactions. We needed only one axially directed support to ensure equilibrium for the stretched (or compressed) bar.

The internal forces $N(x)$ are the force distributions or stress resultants needed to maintain internal equilibrium. Stresses describe the distribution of the internal forces over planar sections drawn through the body's interior. They were defined as point functions of the body's coordinates. So, although it seemed relatively straightforward to define a stress as the quotient $N(x)/A$, where A is the bar's cross-sectional area, we want to extend and generalize this simple definition.

The strains are measures of the deformation of the body that result from the applied forces. There are many definitions of strain, which we reviewed in Section 2.1. The strains are specifically related to the stresses by constitutive laws that describe the properties of the material of which the body is made (cf. Section 2.3). The strains are required to be compatible, by which we mean that their point-by-point variation cannot produce holes in the continuous material of which the body is made, nor can they permit deformation that violates any geometrical constraints relative to the supports that keep the body in place. Simply put, we want our models to reflect "well-behaved" deformation that doesn't produce physically untenable results.

The displacements or the deflections are the (generally) more visible movements of the body. The strains are typically found by differentiating the displacements or deflections with respect to spatial coordinates, as we began to see in Section 2.1.1 and further explore in Chapter 3. The deflections must also be compatible—that is, they must conform with the geometry of the body and its support constraints.

In the language of continuum mechanics, we can now restate our four-item checklist as three major physical considerations that must be applied:

- *Equilibrium* considerations relate external forces, reactions, internal forces, and stresses. That is, we apply Newton's second law to relate external loads to reactions; the method of sections to relate external forces and reactions to internal (resultant) forces; and both Newton's laws and the method of sections to relate internal forces to stresses.
- *Constitutive laws* relate stresses to strains. We invoke constitutive laws to describe the properties of the material of which a body is made.
- *Compatibility* considerations relate strains to displacements or deflections (i.e., kinematics). We pay attention to compatibility both when calculating movements and deflections and when ensuring consistency and continuity with respect to the geometry of the body and its support constraints.

The order in which we apply these criteria, or in which we check off items on our checklist, is not important. The requirement is that our analyses include all of them, no matter the order.

2.13 Examples

Example 2.1

Figure 2.29a shows a diagram of the bones and biceps muscle of a person's arm supporting a mass; Figure 2.29b shows a biomechanical model of the arm, in which the biceps muscle AB is represented by a bar with pin supports. The suspended mass is $m = 2$ kg, and the weight of the forearm is 9 N. If the cross-sectional area of the tendon connecting the biceps to the forearm at A is 28 mm^2, what is the average normal stress in the tendon?

Given: Dimensions of and loading on truss system.

Find: Average normal stress in tendon AB.

Assume: Equilibrium; planar system; neglect weight of muscle and tendon AB.

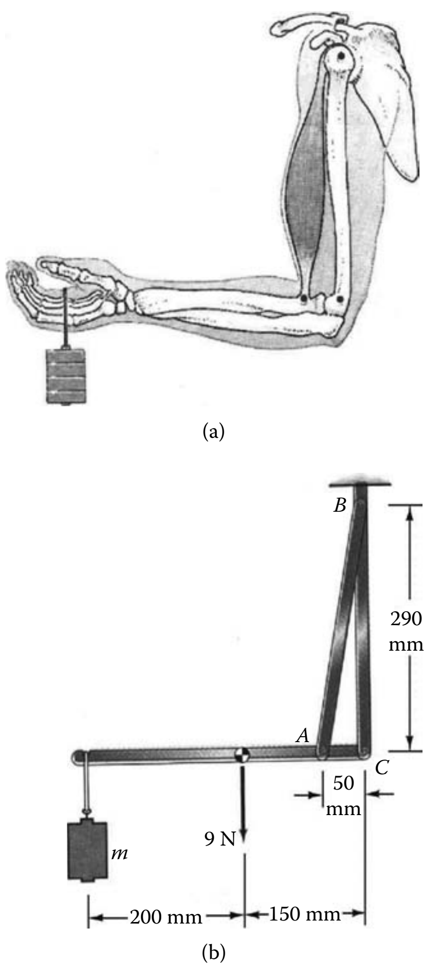


FIGURE 2.29

Solution

We first need to find the internal axial force in AB and then calculate the normal stress by dividing this force by the cross-sectional area. We must construct an FBD of the system (Figure 2.30):

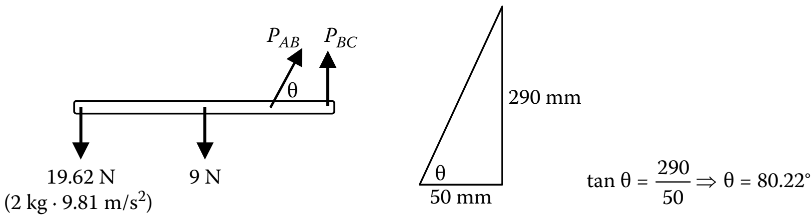


FIGURE 2.30

Equilibrium requires that the sum of moments taken about point *C* be zero, where a counterclockwise moment is taken to be positive:

$$\Sigma M_C = 0 = 19.62\text{ N }(0.35\text{ m}) + 9\text{ N }(0.15\text{ m}) - P_{AB}\sin\theta\ (0.05\text{ m}).$$

Solving for P_{AB},

$$P_{AB} = \frac{8.22\text{N}\cdot\text{m}}{(0.05\text{m})\ \sin\theta} = 166.76\text{ N}.$$

The average normal stress σ_{AB} is then

$$\sigma_{AB} = \frac{P_{AB}}{A_{AB}} = \frac{166.76\text{N}}{28\times 10^{-6}m^2} = 5.95\text{ MPa}.$$

Example 2.2

An infinitesimal rectangle at a point in a reference state of a material becomes the parallelogram shown in a deformed state (Figure 2.31). Determine (a) the extensional strain in the dL_1 direction; (b) the extensional strain in the dL_2 direction; and (c) the shear strain corresponding to the dL_1 and dL_2 directions.

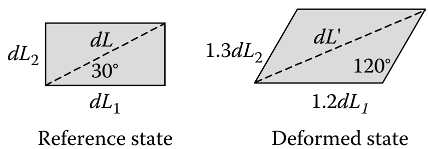


FIGURE 2.31

Given: Reference and deformed geometries of infinitesimal rectangle.

Find: Normal and shear components of strain.

Assume: Strain definitions are adequate; use of "true" strain integral is unnecessary.

Solution

Normal strain in dL_1 direction:

$$\varepsilon_1 = \frac{dL_1{}' - dL_1}{dL_1} = \frac{1.2dL_1 - dL_1}{dL_1} = 0.2.$$

Normal strain in dL_2 direction:

$$\varepsilon_2 = \frac{dL_2{}' - dL_2}{dL_2} = \frac{1.3dL_2 - dL_2}{dL_2} = 0.3.$$

Shear strain is the angular deformation, or change in angle between two reference lines. In reference state, the angle between dL_1 and dL_2 is 90°, or $\pi/2$. In the deformed state, the angle between dL_1' and dL_2' is 60°, or $\pi/3$. The shear strain is thus

$$\gamma_{12} = \frac{\pi}{6} = 0.524 \text{ radians.}$$

Note: If we tried to approximate shear strain by the tangent of this angular deformation instead of using the angle itself, we would get

$$\gamma = \frac{1.3dL_2 \sin \pi / 6}{dL_2} = 0.650 \text{ radians.}$$

This is close, but not *that* close, to 0.524 radians. The angular change in this problem is not sufficiently small to justify the use of the tangent in place of the angle itself.

Example 2.3

Three metal balls are suspended by three wires of equal length arranged in sequence as shown in Figure 2.32. The masses of the balls, starting at the top, are 2 kg, 4 kg, and 3 kg, respectively. In the same order, beginning at the top, the wires have diameters 2 mm, 1.5 mm, and 1 mm, respectively. (a) Determine the highest stressed wire, and (b) by changing the location of the balls, optimize the mass locations to achieve a system with minimum stresses.

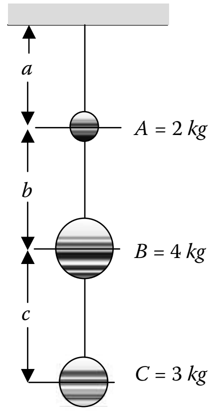


FIGURE 2.32

Given: Dimensions and arrangement of steel balls.

Find: Stresses in each wire; lowest-stress configuration.

Assume: Neglect weights of wires.

Solution

We must find the internal force within each wire and then divide by the wire's cross-sectional area to find the normal stress in each wire. For each wire, the internal force will equal the mass this wire must support times the acceleration of gravity. For example, the top wire, *a*, must support 2 + 4 + 3 kg, so its internal axial force is 88.3 N. We tabulate these calculations:

	P_i (N)	A_i (m^2)	σi (MPa)
Wire a	88.3	3.14×10^{-6}	28.1
Wire b	68.7	1.77×10^{-6}	38.8
Wire c	29.4	0.79×10^{-6}	37.2

The wire subjected to the highest stress is wire *b*.

To achieve a minimum stress system, we recognize that stress is inversely proportional to cross-sectional area. Hence, since $A_a > A_b > A_c$, wire *a* should carry the largest load (which it must), and wire *b* and wire *c* should support as little load as possible. This leads us to the following configuration (Figure 2.33):

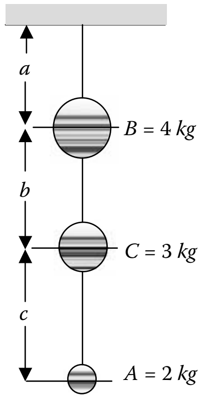


FIGURE 2.33

	σ_i *(MPa)*
Wire *a*	28.1
Wire *b*	27.7
Wire *c*	24.8

In the configuration of part (a), the total stress in the three-wire system is 104 MPa; in part (b), the total stress is 80.6 MPa.

Example 2.4

A steel bar 10 m long used in a control mechanism must transmit a tensile force of 5 kN without stretching more than 3 mm or exceeding an allowable stress of 150 MN/m^2. What must the diameter of the bar be? State your answer to the nearest millimeter, and use E = 200 GPa.

Given: Dimensions and loading on steel bar.

Find: Required bar diameter to nearest mm.

Assume: Hooke's law applies.

Solution

We will impose both *strength* and *stiffness* constraints on the bar and will see which is the limiting case.

Using the definition of normal stress, we must have

$$\sigma = \frac{P}{A} \leq 150 \frac{\text{MN}}{\text{m}^2},$$

or

$$A \geq \frac{P}{150\ \text{MN/m}^2} = \frac{5000\ \text{N}}{150 \times 10^6 \text{N/m}^2} = 33.33 \times 10^{-6}\ \text{m}^2,$$

that is,

$$A \geq 33.33\ \text{mm}^2.$$

If Hooke's law applies, as we have assumed it does, then

$$\delta = \frac{PL}{AE},$$

and we must have

$$\frac{PL}{AE} \leq 3\ \text{mm}$$

or

$$A \geq \frac{PL}{\delta\, E} = \frac{5000\ \text{N} \cdot\ 10\ \text{m}}{(0.003\ \text{m})(200 \times 10^9 \text{N/m}^2)} = 83.33 \times 10^{-6}\ \text{m}^2$$

that is,

$$A \geq 83.33\ \text{mm}^2.$$

We see that stiffness is the limiting case and that we must have a cross-sectional area greater than or equal to 83.33 mm^2 to safely meet our constraint. This is all we need to find the required diameter of the steel bar:

$$\frac{\pi}{4} d^2 \geq 83.33\ \text{mm}^2,$$

so

$$d \geq 10.3\ \text{mm}.$$

So to the nearest millimeter, we must use an 11-mm-diameter bar.

Example 2.5

A solid bar 50 mm in diameter and 2000 mm long consists of a steel and an aluminum section, as shown in Figure 2.34. When axial force P is applied to the system, a strain gauge attached to the aluminum indicates an axial strain of 0.000873 m/m. (a) Determine the magnitude of applied force *P*, and (b) if the system behaves elastically, find the total elongation of the bar.

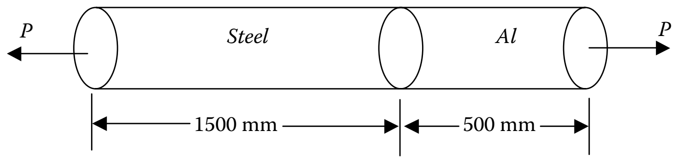


FIGURE 2.34

Given: Dimensions of composite bar and measured normal strain.

Find: Applied force, *P*, and elongation of bar, δ.

Assume: Hooke's law applies.

Solution

The diameter of the bar is 50 mm, so the cross-sectional areas of both parts are equal:

$$A_{St} = A_{Al} = \frac{\pi}{4}\left(0.05\text{ m}\right)^2 = 0.00196\text{ m}^2.$$

The elastic moduli for aluminum and steel may be looked up in Table 2.1 or in another reference.

$$E_{Al} \approx 70\text{ GPa, and } E_{St} \approx 200\text{ GPa.}$$

If Hooke's law applies, we can relate the strain measured in the aluminum portion to the stress induced by *P* in that portion:

$$\varepsilon_{Al} = 0.000873 = \frac{\sigma_{Al}}{E_{Al}} = \frac{(P / A_{Al})}{E_{Al}},$$

so

$$P = (0.000873)(70 \times 10^9\text{ Pa})(0.00196\text{ m}^2) = 120\text{ kN.}$$

We can exploit Hooke's law and superpose the displacements of both portions of the bar:

$$\delta = \sum \frac{PL}{AE} = \left(\frac{PL}{AE}\right)_{St} + \left(\frac{PL}{AE}\right)_{Al}$$

$$= \frac{120{,}000\ \text{N}}{0.00196\ \text{m}^2}\left[\frac{1.5\ \text{m}}{200\times10^9\ \text{N/m}^2}+\frac{0.5\ \text{m}}{70\times10^9\ \text{N/m}^2}\right]$$

$$= 459 \times 10^{-6}\ \text{m} + 437 \times 10^{-6}\ \text{m} = 896 \times 10^{-6}\ \text{m}$$

$$\delta = 896\ \mu\text{m, or } 0.896\ \text{mm.}$$

Note: The aluminum section is only a third as long as the steel, but it deforms nearly as much!

Example 2.6

A polystyrene bar consisting of two cylindrical portions AB and BC is restrained at both ends and supports two 26 kN loads as shown in Figure 2.35. Knowing that E is 3.1 GPa, determine (a) the reactions at A and C, and (b) the normal stress in each portion of the bar.

Given: Dimensions of and loading on composite polystyrene bar.

Find: Reactions and normal stresses.

Assume: Hooke's law applies. Neglect weight of polystyrene cylinders.

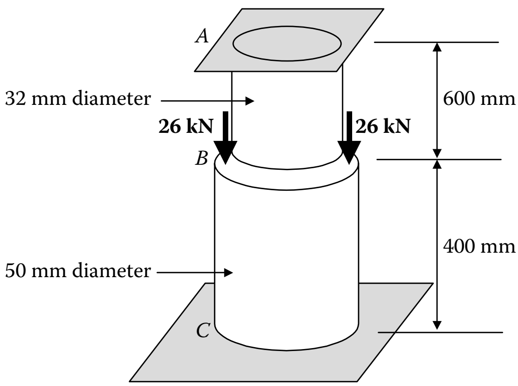


FIGURE 2.35

Solution

The first thing we need is an FBD (Figure 2.36):

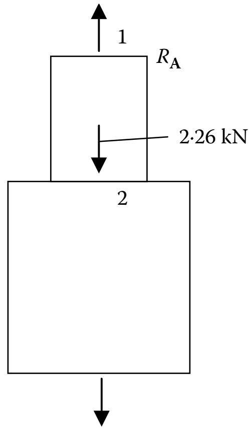


FIGURE 2.36

We ensure that this system is in equilibrium by stating,

$$\Sigma F_y = 0, \text{ or } R_A + R_B = 52 \text{ kN}.$$

This one equation contains two unknowns: The problem is statically indeterminate. So, what else do we know? Because both ends are fixed, the total elongation of the composite bar must be zero. So,

$$\delta_{AB} + \delta_{BC} = 0.$$

Using Hooke's law, we can write these displacements as

$$\delta_{AB} = \frac{P_{AB}L_{AB}}{A_{AB}E} \text{ and } \delta_{BC} = \frac{P_{BC}L_{BC}}{A_{BC}E}.$$

We then use the method of sections to find P_{AB} and P_{BC}, the internal forces in the two component sections (Figure 2.37).

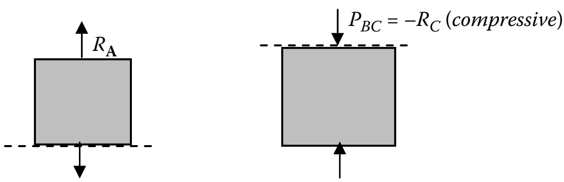


FIGURE 2.37
Measurement of tension in a string.

So, imposing the geometric constraint that the total elongation of the bar is zero, we have

$$\frac{R_A L_{AB}}{A_{AB}E}+\frac{-R_C L_{BC}}{A_{BC}E}=0,$$

or

$$(2.407 \times 10^{-7})R_A - (6.572 \times 10^{-7})R_C = 0.$$

We also know $R_A + R_C = 52$ kN, so we can solve these two equations to get the reaction forces:

$$R_A = 11.2 \text{ kN}$$

$$R_C = 40.8 \text{ kN}.$$

Then we divide the internal forces by the cross-sectional areas they act on to obtain the normal stresses in both pieces:

$$\sigma_{AB}=\frac{P_{AB}}{A_{AB}}=\frac{R_A}{A_{AB}}=\frac{11.2\times10^3 \text{ N}}{\frac{\pi}{4}(0.032 \text{ m})^2}=14.0 \text{ MPa}$$

$$\sigma_{BC}=\frac{P_{BC}}{A_{BC}}=\frac{-R_c}{A_{BC}}=\frac{-40.8\times10^3 \text{ N}}{\frac{\pi}{4}(0.05 \text{ m})^2}=-20.8 \text{ MPa}.$$

Example 2.7

Each bar in the truss shown in Figure 2.38 has a 2 in.^{2} cross-sectional area, modulus of elasticity $E = 14 \times 10^6$ psi, and coefficient of thermal expansion $\alpha = 11 \times 10^{-6}$ (°F)$^{-1}$. If their temperature is increased by 40°F from their initial temperature T, what is the resulting displacement of point A? What upward force must be applied to prevent this displacement?

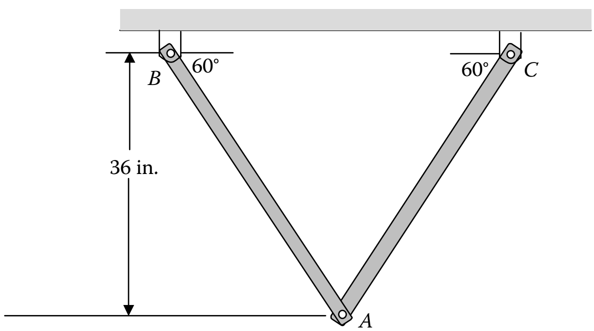


FIGURE 2.38

Given: Dimensions and properties of truss; imposed temperature change.
Find: Displacement of point *A*; force necessary to prevent this displacement.
Assume: Hooke's law applies. Neglect weight of bars.

Solution

The geometry of the problem allows us to find the original length of bars *AB* and *AC* (Figure 2.39):

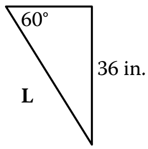


FIGURE 2.39

$$L = 36\text{ in.}/\sin(60°) = 41.56\text{ in.}$$

The change in the length of each bar due to the change in temperature ΔT is then

$$\delta_T = L\,\alpha\,\Delta T = (41.56\text{ in.})(11 \times 10^{-6}\,(°\text{F})^{-1})(40°\text{F}) = 0.018\text{ in.}$$

So, the new vertical distance from the fixed surface to point *A* is

$$(41.56 + 0.018)\cdot \sin\,(60^\circ) = 36.008\text{ in.}$$

The horizontal displacements of *AB* and *AC* will be equal and opposite, so the net displacement of point *A* is only vertical and is 0.008 in.

The upward force applied to prevent this must "undo" the thermal expansion of the two bars; it must induce a compressive axial load *P* in both bars such that, by Hooke's law,

$$\frac{PL}{AE} = -0.018\text{ in., so } P = \frac{(-0.018\text{ in.})(14\times 10^6\text{ psi})(2\text{ in.}^2)}{41.56\text{ in.}} = -12{,}127\text{ lb (compressive!)}.$$

We construct an FBD and use equilibrium to find the force *F* necessary to induce this compressive load *P* in both bars (Figure 2.40):

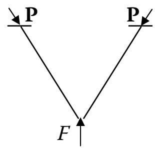


FIGURE 2.40

$$\Sigma F_y = 0 = F - 2P\sin(60^\circ), \text{ or } F = 2P\sin(60^\circ)$$
$$F = 21{,}327 \text{ lb.}$$

Example 2.8

A steel railroad track (E = 200 GPa, $\alpha = 11.7 \times 10^{-6}/^\circ\text{C}$) was laid out at a temperature of 0°C. Determine the normal stress in a rail when the temperature reaches 50°C, assuming that the rails are (a) welded to form a continuous track, or (b) 12 m long with 6-mm gaps between them.

Given: Geometry of problem, material properties, imposed temperature change.

Find: Normal stress (a) when continuous or (b) when gaps are left.

Assume: Hooke's law applies.

Solution

Based on our understanding of thermal stresses, we expect the stress calculated in part (b) to be lower than that in part (a): We have learned that thermal stresses are induced only when a part is prevented from experiencing its natural thermal deformation, so the space left to accommodate thermal expansion in part (b) should help relieve the induced stress. We will see whether this expectation is met.

A schematic helps to illustrate the problem (Figure 2.41):

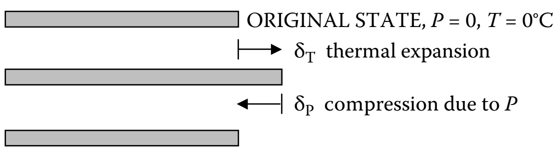


FIGURE 2.41

(a) The total deformation of a steel track segment is $\delta_T + \delta_P = 0$, as the welding allows no net change to the length of the segments. Hence, we add the deformations due to thermal effects and compressive forces:

$$0 = \underbrace{\alpha\,(\Delta T)L}_{\text{tends to stretch}} + \underbrace{\frac{-PL}{AE}}_{\text{tends to squash}},$$

so

$$\alpha\,\Delta T = \frac{P}{AE} = \frac{\sigma}{E}$$

$$\sigma = \alpha\,\Delta T\,E = (11.7 \times 10^{-6}\,(^{\circ}\mathrm{C})^{-1})(50 - 0^{\circ}\mathrm{C})(200 \times 10^{9}\ \mathrm{Pa})$$

$$\sigma = 117\ \mathrm{MPa\ (compressive)}$$

when welded.

(b) If a gap of 6 mm is left between rails, we allow each segment a net stretch of 6 mm:

$$+0.006\ \mathrm{m} = \alpha\,(\Delta T)L - \frac{PL}{AE},$$

so

$$\sigma = \frac{\alpha\,\Delta\,TL - 0.006\ \mathrm{m}}{L}E = \left(\frac{11.7\times10^{-6}(^{\circ}\mathrm{C})^{-1}(50^{\circ}\mathrm{C})(12\ \mathrm{m}) - 0.006\ \mathrm{m}}{12\ \mathrm{m}}\right)\left(200\times10^{9}\ \mathrm{Pa}\right)$$

$$\sigma = 17\ \mathrm{MPa\ (compressive)}$$

when a gap is left.

Example 2.9

A 6 mm × 75 mm plate, 600 mm long, has a circular hole of 25 mm diameter located at its center. Find the axial tensile force that can be applied to this plate in the longitudinal direction without exceeding an allowable stress of 220 MPa. How does the presence of the hole affect the strength of the plate?

Given: Dimensions of plate, limiting normal stress.

Find: Allowable axial load that can be applied to plate.

Assume: Hole is only feature that causes a stress concentration.

Solution

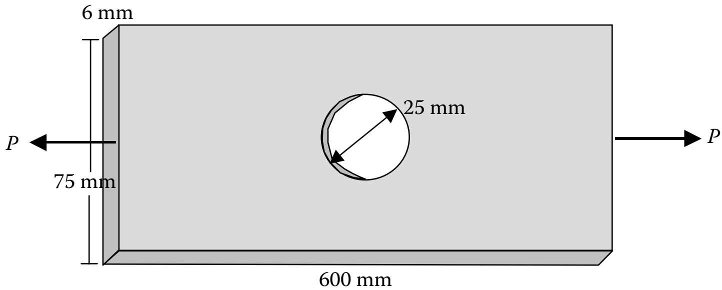


FIGURE 2.42

The cross-sectional area normal to an axial load P is $A_o = 6\text{ mm} \times 75\text{ mm} = 450\text{ mm}^2$ (Figure 2.42). The average normal stress induced by such a load will be

$$\sigma_{ave} = P/A_o,$$

and due to the presence of the hole we must consider the effects of *stress concentration:*

$$\sigma_{\max} = K\sigma_{ave} = K\frac{P}{A_o}.$$

We can find K for this geometry using the graph in Figure 2.27:

$$\frac{r}{d} = \frac{(25\text{ mm})/2}{75\text{ mm} - 25\text{ mm}} = \frac{1}{4},$$

so

$$K(\tfrac{r}{d} = 0.25) = 2.26$$

$$\sigma_{\max} = K\frac{P}{A_o} = 2.26\frac{P}{450\text{ mm}^2} = 0.005P$$

and since we must have

$$220\text{ MPa} \geq 0.005\,P$$

then

$$P \leq 43.8\text{ kN}.$$

Note: If there were no hole in this plate, we would simply have

$$\sigma_{ave} = P/A_o,$$

and we could allow a force

$$P \leq 99 \text{ kN}.$$

So with the hole, we can permit only 44% of the load we could have allowed without the hole.

2.14 Problems

2.1 In tissue engineering, biological materials are grown from seeded cells, so that artificial corneas, blood vessels, or other materials may be made from biological materials. Such materials are less likely than artificial parts made of plastic or metal to be rejected by the body. To engineer true replacement parts, it is necessary to understand the behavior of physiological systems and to match material properties such as elastic and shear moduli. It is impractical to construct a tension specimen like that in Figure 2.2 from soft tissues such as muscles, tendons, or blood vessels. What would you do instead?

2.2 Concrete, rocks, and bone are strong in compression and are usually designed for compressive loading. To test their strength in compression, what sort of test specimen would be useful?

2.3 The tension in your Achilles tendon is considerable when you stand on tiptoe or poise for a jump. Design a tension gauge that might be useful in measuring such tension, or the tension in a bow string or rubber slingshot. (*Hint*: after Fung 1994; Figure 2.43):

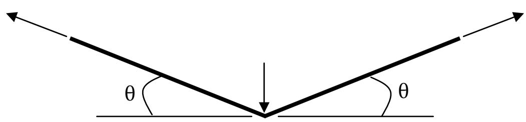


FIGURE 2.43

2.4 When a kangaroo switches from "pentapedal" (four limbs and tail) locomotion to hopping, its oxygen consumption drops, presumably because it then stores more energy in elastic tissues (Dawson and Taylor 1973). One of these elastic tissue "springs" in kangaroos (and other animals) is the Achilles tendon. A kangaroo's Achilles tendon was found to be 1.5 cm in diameter and 35 cm long. If each Achilles tendon has an elastic modulus of 1 GPa and is loaded to 2% strain (below its proportional limit), how much strain energy (i.e., stored potential energy) would both Achilles tendons contain? Based strictly on energy considerations, can you predict how high this amount of energy could lift a 40 kg kangaroo?

2.5 In Figure 2.44, the suspended mass $m = 20$ kg. Determine the axial force in the bar AB, and indicate whether it is in tension or compression. (*Hint*: Draw the free-body diagram of joint B.)

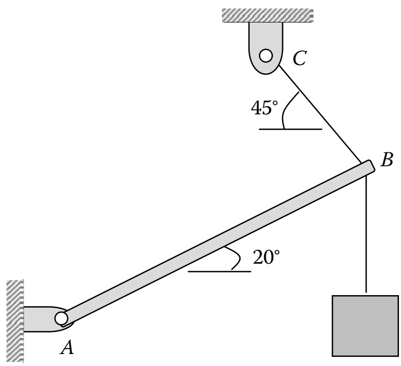


FIGURE 2.44

2.6 The bar shown in Figure 2.46 has a solid circular cross section, with a 2 in. radius. Determine the average normal stress (a) at plane P_1, and (b) at plane P_2.

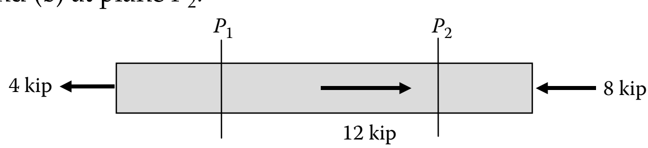


FIGURE 2.45

2.7 Suppose that a downward force is applied at point A of the truss, causing point A to move 0.360 in. downward and 0.220 in. to the left (Figure 2.46). If the resulting extensional strain ε_{AB} in the direction parallel to the axis of bar AB is uniform, what is ε_{AB}?

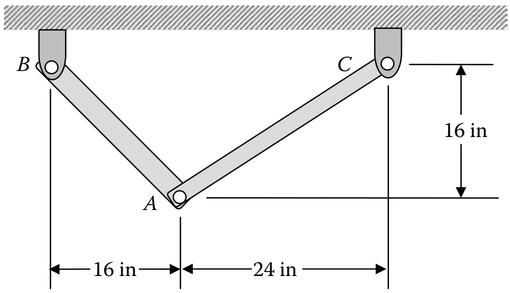


FIGURE 2.46

2.8 The jaws of the bolt cutter shown in Figure 2.48 are connected by two links *AB*. The cross-sectional area of each link is 750 mm^2. (a) What average normal stress is induced in each link by the 90 N forces exerted on the handles? (b) The pins connecting the links *AB* to the jaws of the bolt cutter are 20 mm in diameter. What average shear stress is induced in the pins by the 90 N forces exerted on the handles?

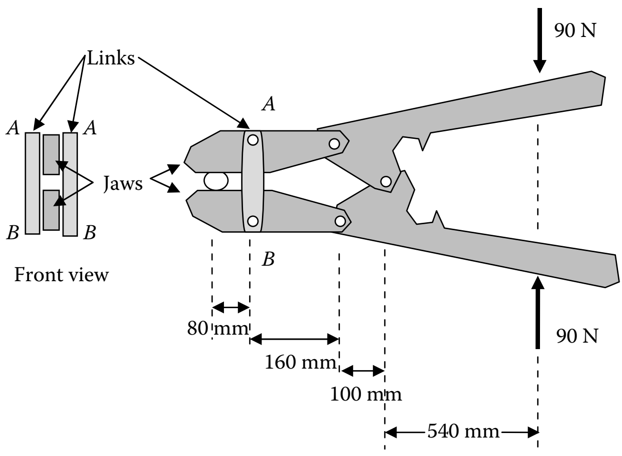


FIGURE 2.47

2.9 Determine the maximum allowable value of the force *F* if the tensile stress in segment *AB* must be less than 150 MN/m^2 (Figure 2.48). What are the changes in length of segment *BC* and of the entire bar for this value of *F*? The bar's cross-sectional area is 50 mm^2, and the bar is made of steel.

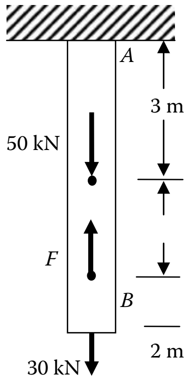


FIGURE 2.48

2.10 The bar shown in Figure 2.49 has a constant cross section and is fixed rigidly at both walls. Determine the reactions at both walls for the given applied load P.

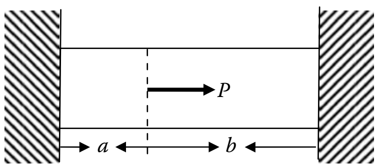


FIGURE 2.49

2.11 A rigid slab with mass m = 15,000 kg is supported by three columns, as shown in Figure 2.50. Determine the compressive force in each of the columns.

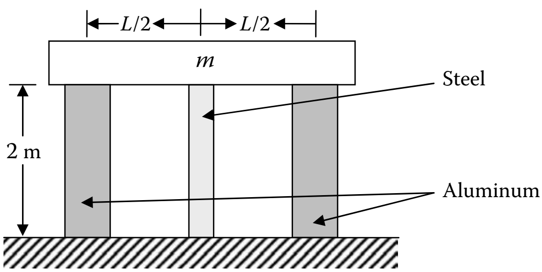


FIGURE 2.50

2.12 The bar shown in Figure 2.51 has a varying cross section and is fixed rigidly at both walls. The cross-sectional area of the narrower section is A; the cross-sectional area of the wider section is larger by a factor of m, or mA. Using the force (flexibility) method, determine (a) the reactions at both walls for the given applied load P; and (b) the displacement of the point D at which the load P acts.

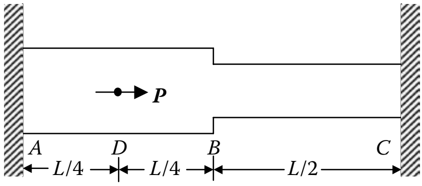


FIGURE 2.51

2.13 The bar shown in Figure 2.51 has a varying cross section and is fixed rigidly at both walls. The cross-sectional area of the narrower section is A; the cross-sectional area of the wider section is larger by a factor of m, or mA. Using the displacement (stiffness) method, determine (a) the reactions at both walls for the given applied load P; and (b) the displacement of the point D at which the load P acts.

2.14 Determine the movement δ of the tip of a dense, heavy uniform bar hanging vertically from one end. The bar has length L, modulus E, cross-sectional area A, and mass density ρ.

2.15 Your local lumber yard is providing a set of wooden 4 in. × 4 in. posts that you will mount on 6 in. × 6 in. concrete bases to support a section of roof. Handbooks provide the allowable compressive stresses: 1800 psi for wood and 1250 psi for concrete. The specific weight of concrete is also given as 150 lbf/ft^3, although the corresponding number for wood is not shown. For the postsupport configuration shown in Figure 2.52, determine (a) the specific weight of wood, given that a 2 ft section of the post weighs 21 lb and knowing that a 4 in. × 4 in. post is actually 3.5 in. × 3.5 in.; (b) the supported load P when the weights of the support and post are included; and (c) the supported load P when the weights of the support and post are not included.

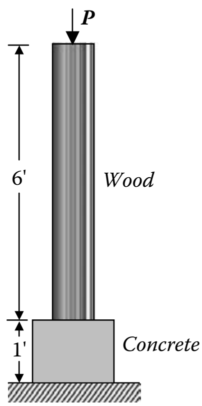


FIGURE 2.52

2.16 For the wood block shown in Figure 2.53, the allowable shear stress parallel to the grain is 1 MN/m^2, and the maximum allowable compressive stress in any one direction is 4 MN/m^2. Determine the maximum compressive force F that the block can support.

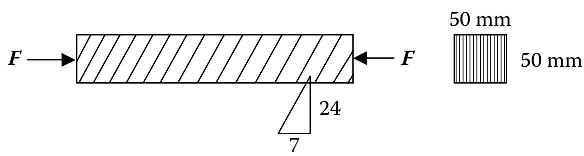


FIGURE 2.53

2.17 Two cylindrical bars with 30 mm diameters, one (*ABC*) made of yellow brass and the other (*CDE*) of stainless steel, are joined at *C* (Figure 2.54). End *A* of the composite bar is fixed, while there is a gap of 0.2 mm between end *E* and a vertical wall. A force of magnitude 40 kN and directed to the right is applied at *B*. Determine (a) the smallest force *P* needed at *D* to just close the gap without the steel bar actually coming into contact with the wall at *E;* (2) the reactions at *A* and *E* if a 40 kN force directed to the right is applied at *D;* and (3) the reactions at *A* and *E* if force *P* is twice the value you calculated in (a).

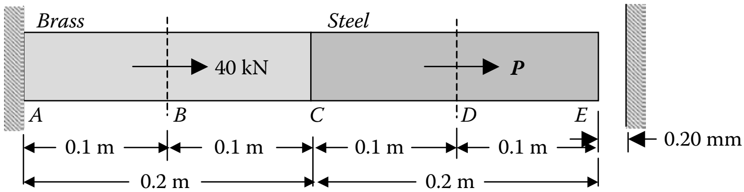


FIGURE 2.54

2.18 A bar consists of two portions BC and CD of the same material and same length L but of different cross sections (Figure 2.55). Determine the strain energy of the bar when it is subjected to an axial load P, expressing the result in terms of P, L, E, the cross-sectional area A of portion CD, and the ratio n of the two diameters.

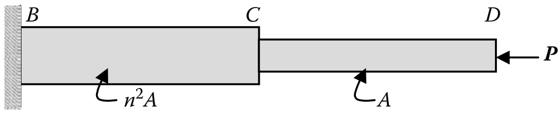


FIGURE 2.55

2.19 An electronic scoreboard is to be installed in a large stadium. Due to the design of the roof structure, the suspending cables will have different lengths, as is shown in Figure 2.56. (a) Determine a suitable cross-sectional area for each cable so that the scoreboard will hang level, accounting for the stretch in each cable. Use the data in the figure and the requirement that the cable's yield strength is 36 ksi. The modulus of elasticity for the cables is $E = 30{,}000$ ksi, and the weight of the scoreboard is $W = 10$ k. Remember, 1 kip (k) = 1000 lb. (b) The slope of the grain in the longer support cable has a maximum deviation from the cable's longitudinal axis of 15°, and there is some concern that the relatively low shear strength of the cable material along its grain could cause problems. Calculate both normal and shear stresses along this grain.

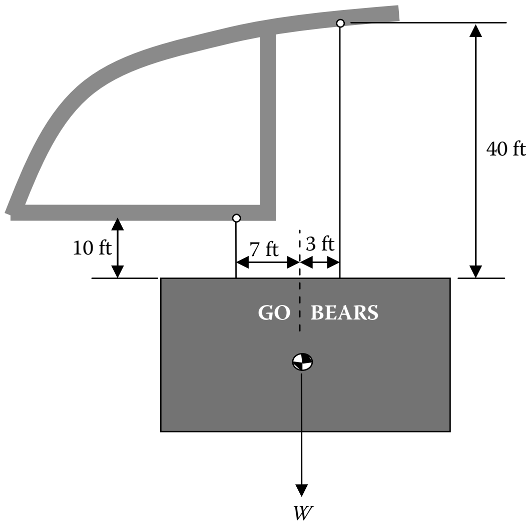


FIGURE 2.56

Case Study 1: Collapse of the Kansas City Hyatt Regency Walkways

On July 17, 1981, in the most damaging unforced structural failure in the history of the United States, two overhead walkways fell into the atrium lobby of the Hyatt Regency Hotel in Kansas City, Missouri. As a result of this collapse 114 people died, and millions of dollars of damage was sustained (Figure CS1.1).

The failure derived in large part from a key aspect of modern engineering design, which is that engineering designers do not, typically, build what they design. Rather, they produce a *fabrication specification*, a detailed description of the designed object that allows its assembly or manufacture by others. Separating the "designing" from the "making" means that such fabrication specifications must be complete and unambiguous.

Fabrication specifications are presented in drawings (e.g., blueprints, circuit diagrams, flow charts) and in text (e.g., parts lists, materials specifications, assembly instructions). Such traditional specifications can be complete and sufficiently specific, but they may not capture the designer's intent—and this can lead to catastrophe. The suspended walkways in the Hyatt Regency Hotel in Kansas City collapsed because a contractor fabricated the connections for the walkways in a manner different from the original design.

FIGURE CS1.1
The lobby of the Kansas City Hyatt Regency Hotel after the collapse of the second- and fourth-floor walkways on July 17, 1981. The devastation is evident. (Courtesy of Lee Lowry, Kansas City, MO.)

In the original design, walkways at the second and fourth floors were hung from the same set of 24-ft-long threaded rods that would carry their weights and loads to a roof truss (Figure CS1.2). The fabricator was unable to procure threaded rods sufficiently long to suspend the second-floor walkway from the roof truss, so instead, as shown in Figure CS1.3, he hung it from the fourth-floor walkway using shorter rods. (The original design would not have been easy to implement because of the difficulty involved in screwing on bolts over such long hanger rods and attaching walkway support beams.) The supports of the fourth-floor walkway were not designed to carry both the second-floor walkway and its own dead and live loads, resulting in the collapse. If the fabricator had understood the designer's intention to hang the second-floor walkway directly from the roof truss, this accident might have been avoided.

As Henry Petroski (1982) noted, the fabricator's redesign was akin to requiring that the lower of two climbers hanging independently from the same rope change his position so that he was grasping the feet of the climber above, causing the upper climber to carry the weights of both with respect to

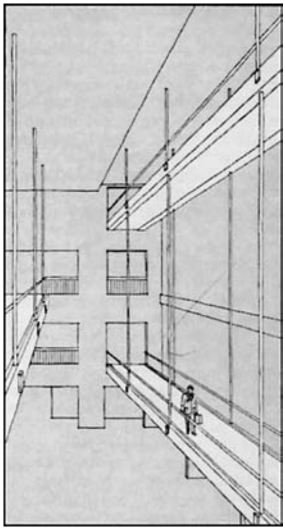


FIGURE CS1.2
An artist's sketch of the second- and fourth-floor walkways across the west side of the atrium of the Kansas City Hyatt Regency Hotel. The view looks southward and also shows a separate third-floor walkway that did not collapse but was taken down after a design review prompted by the collapse of the other two walkways on July 17, 1981. (From Pfrang, E. O. and Marshall, R., with E. J. Orwin and R. E. Spjut, *Civil Engineering*, pp. 65–68, July 1982. With permission.)

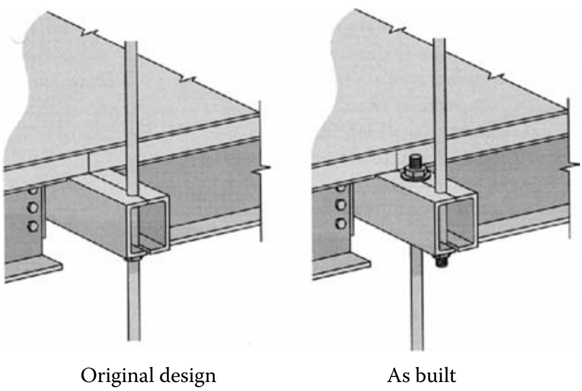


FIGURE CS1.3
The two hanger connections at the fourth-floor walkway: (a) The left sketch shows the configuration as designed, wherein the hanger rods went straight through the fourth-floor connection, down to the second floor, which these rods also supported. (b) The right sketch shows the configuration as built, with the hanger rods supporting the second floor now hung from the box beams that hold up the fourth-floor walkway. (From Dym, C. L. and Little, P., *Engineering Design: A Project-Based Introduction*, 3rd Ed., John Wiley & Sons, New York, 2008. With permission.)

the rope. The redesigned supports for the second-floor walkway were configured similarly.

Figure CS1.4 shows several sketches of the original design: (a) an elevation view of the second- and fourth-floor walkways, *each supported by the same pairs of hanger rods* (on east and west sides of the walkway) spaced at a distance L; and (b) an end view of the two walkways and FBDs of the supporting beam of each walkway. Consider now the lower, second-floor walkway. The load carried by each pair of its hanger rods can be estimated as the sum of the *dead load* of the walkway and its supporting beams and the *live load* of pedestrians likely to stand or walk across the walkways. Since the hanger rods are spaced a distance L apart we estimate the total force $2P$ needed to support a span of length $L/2$ on either side of a pair of hangers as

$$2P = (w + W)bL, \tag{CS1.1}$$

where w is the dead load per unit area, W the live load per unit area, and b the walkway width. In this instance, by both making calculations based on the design drawings and weighing pieces of the collapsed walkways, the engineers at the National Bureau of Standards (NBS)[9] who performed the forensic investigation of the walkway collapse determined that the combination of the dead and live loads, called the *design load*, was in this case $P = 90$ kN (20,300 lbf) per hanger rod. The analysis of the fourth-floor walkway based on the original design would be the same. Then the individual hanger rods needed to support both the second- and fourth-floor walkways as designed would each support a total load of $2P$ and would be sized accordingly.

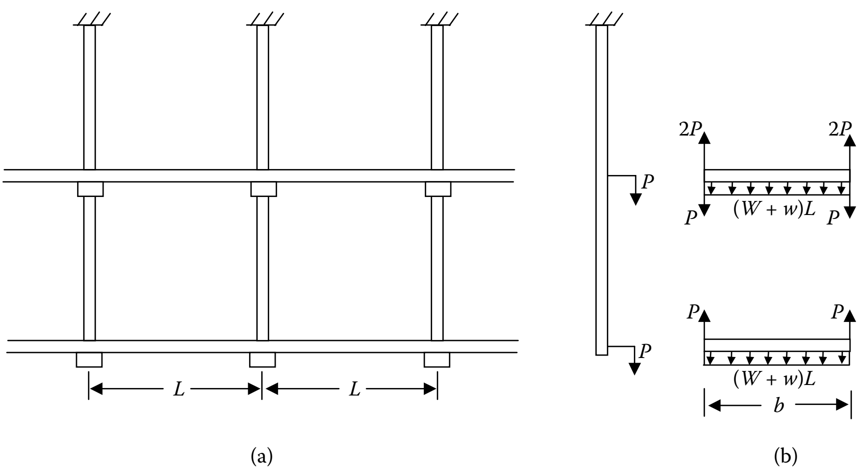


FIGURE CS1.4
Building a model of the walkways and their supports: (a) An elevation of the second- and fourth-floor walkways as originally designed. (b) An end view and free-body diagrams of the support beams. The forces carried by the hanger rods accumulate according to the number of walkways being supported below them.

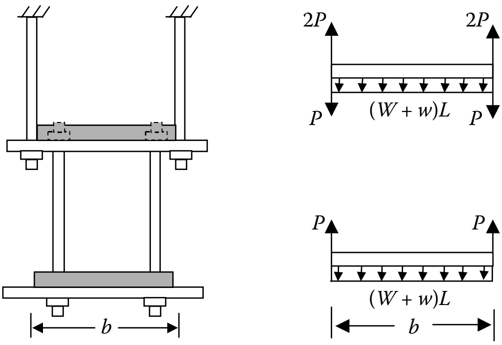


FIGURE CS1.5
Extending the model of the walkways and their supports to reflect the redesign. An end view of the second- and fourth-floor walkways designed so that the second-floor walkway hangs from the fourth-floor supporting beams, and free-body diagrams of a typical pair of supports. Note that the forces supported by the hanger rods are unchanged from the original design.

On the other hand, the end views of the walkways as built and their corresponding FBDs (Figure CS1.5) show that the rods would have to carry exactly the same loads at each level; that is, to support the lower walkway each rod carries a load equal to P, while above the fourth floor each rod would have to carry a load of $2P$ to support both the second- and fourth-floor walkways. So the rods in both designs would have equivalent designs with the same area, determined by equation (2.7),

$$A = \frac{2P}{\sigma_{\text{allow}}}, \tag{CS1.2}$$

where σ_{allow} is the allowable stress in the rod. In terms of the rope analogy, the part of the rope above the two climbers has to support the weight of both: It doesn't care whether each hangs directly from the rope or one climber hangs from the other.

So, why did the walkways collapse? They failed because an unanticipated connection was inserted into the design, the connection was not properly designed, and it failed (Figure CS1.6). As noted by respected engineers E. O. Pfrang and R. Marshall (1982), "With this modification the design load to be supported by each second floor … connection was unchanged …. However, the load to be transferred from the fourth floor … to the upper hanger rod under this arrangement was essentially doubled" (p. 68). Look again at the FBD in Figure CS1.5: It shows that the redesign required the nut under the fourth-floor supporting beam and its connection with the beam itself to support the transfer of twice the load that would have been transferred in the original design—which the fabricator's redesigned connection did not.

FIGURE CS1.6
Photographs of the failed connections that led to the collapse of the two walkways in the Kansas City Hyatt Regency Hotel. Compare it with Figure CS1.3 (b) and see that the outboard connection (on the right-hand edge) failed because the threaded nut and washer that went underneath the box beam pulled right through the box beam because that connection, designed originally to transmit a load of *P*, was actually carrying a load of 2*P*. (Courtesy of Lee Lowry.)

Interestingly enough, it was also revealed in the subsequent forensic investigation that even the original design was only marginally safe. The NBS investigators found that the long-rod design would likely not have satisfied the Kansas City Building Code specifications. Further, it turned out that during construction, the building's construction workers had noticed that the walkways seemed flimsy and that they moved noticeably whenever workers moved wheelbarrows or the like across them. Their solution? Rather than report the problem and request a fix, they found other routes over which to transport their building materials!

The NBS official report issued in 1981 did not assign blame for this catastrophe. The essential problem was a lack of proper communication between the design engineers (Jack D. Gillum and Associates) and the manufacturers (Havens Steel). However, the NBS report's authors, Pfrang and Marshall, made it clear that responsibility lay primarily with the structural engineers. The Missouri licensing board and Court of Appeals agreed, finding that the design engineers should have noticed the difference between their design and what the contractor suggested and should have analyzed the redesigned connection. Basic calculations should have demonstrated the flaws in both the original design and in what was ultimately built. The principal structural engineers lost their Missouri engineer's licenses, and the firm, Jack D. Gillum and Associates, dissolved. The Hyatt Regency Crown Center lobby in Kansas City today features only one walkway, which is not suspended from the roof but instead rests on sturdy-looking columns that transmit its loads to the atrium floor.

Problems

CS1.1 If the Kansas City Building Code specified that a floor structure must support a live load of 4.79 kPa (100 psf), and if the walkway length $L = 9.1$ m = 30.0 ft and width $b = 2$ m = 6.56 ft, what contribution is made to the hanger rod load P?

CS1.2 If the design load is 90 kN (20,300 lbf), what is the dead load and what is the intensity of the dead load in the light of the live load calculation of Problem CS1.1?

CS1.3 Determine the specific weight of lightweight concrete and calculate its dead load intensity if it is used in an 80-mm (3.25 in.) cover of a formed steel deck walkway. Compare this result with that found in Problem CS1.2 and explain any differences.

CS1.4 Determine the stress induced in hanger rods carrying a design load of 90 kN (20,300 lbf), if their diameters are 32 mm (1.26 in.). Does that seem a reasonable stress level if the rods are made of mild steel? Explain your answer.

CS1.5 If the interfloor distance of the Hyatt Regency Hotel is 4.57 m (15 ft), how much does the second-floor walkway move with respect to the fourth-floor walkway?

Notes

1. Or, more generally, that Newton's second law is satisfied.
2. Carlson was a civil engineer investigating California dams; Simmons was an electrical engineer who first developed a way to manufacture a bonded-wire strain gauge and patented his design—though it took an 11-year court battle for him to win the patent rights for himself and not for Caltech, where he had been educated and continued to work.
3. Temperature considerations are important because the wires' electrical properties may be temperature dependent and also because temperature itself can result in deformation, as is quantified in Section 2.9.
4. Both P and F are used to represent forces in this textbook. The vector A is nA, where n is the outward normal vector of the area A.
5. From Gordon (1988) p. 45.
6. Such experiments were performed by Jacob Bernoulli (1654–1705) and J. V. Poncelet (1788–1867) in the quest to understand materials' response to loading.
7. Hooke (1635–1703) has never received due recognition for his scientific achievements. In addition to crafting what we know as Hooke's law, Hooke was an architect and geologist whose studies of microorganisms (using his friend Anton van Leeuwenhoek's newfangled microscopes) and fossils were seminal. Hooke's relative obscurity is largely a result of the efforts of his vindictive contemporary, Sir Isaac Newton, who used his own fame and influence to

diminish Hooke's accomplishments; it was his fear that Newton would steal or diminish *ut tensio, sic vis* that led Hooke to publish only his encrypted anagram for Hooke's law. He and Newton had had a feud over the inverse-square law of planetary motion, and Newton was so perturbed by it that he removed all traces of Hooke from his *Principia*. Hooke had even been prescient enough to anticipate the application of his observation of springs to material behavior, having stated that every kind of solid changes its shape when a mechanical force is applied and that it is this deformation that enables the solid to do what Gordon (1988) called "the pushing back." In so observing, Hooke anticipated the fields of continuum mechanics and elasticity. However, his intellectual heirs Thomas Young and Leonhard Euler were denied their inheritance by Newton, and Hooke remained obscure. Furthermore, we do not know what Hooke looked like, perhaps because—as he is often described as a "lean, bent, and ugly man"—he was unwilling to sit for a portrait.

8. If, instead, equilibrium is expressed in terms of displacements, the distinction between determinate and indeterminate problems vanishes, and we may apply the solution method developed in Section 2.7. However, it may prove useful (cf. Problem 2.8 and Problem 2.9) to work out a technique to resolve the indeterminacy of problems expressed in terms of stresses.
9. Since 1988 called the National Institute of Science and Technology (NIST).